A Method of Constructing Off–Diagonal Solutions in Metric–Affine and String Gravity

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Abstract

The anholonomic frame method is generalized for non–Riemannian gravity models defined by string corrections to the general relativity and metric–affine gravity (MAG) theories. Such spacetime configurations are modeled as metric–affine spaces provided with generic off–diagonal metrics (which can not be diagonalized by coordinate transforms) and anholonomic frames with associated nonlinear connection (N–connection) structure. We investigate the field equations of MAG and string gravity with mixed holonomic and anholonomic variables. There are proved the main theorems on irreducible reduction to effective Einstein–Proca equations with respect to anholonomic frames adapted to N–connections. String corrections induced by the antisymmetric H–fields are considered. There are also proved the theorems and criteria stating a new method of constructing exact solutions with generic off–diagonal metric ansatz depending on 3-5 variables and describing various type of locally anisotropic gravitational configurations with torsion, nonmetricity and/or generalized Finsler–affine effective geometry. We analyze solutions, generated in string gravity, when generalized Finsler–affine metrics, torsion and nonmetricity interact with three dimensional solitons.

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Contents

1 Introduction

Nowadays, there exists an interest to non–Riemannian descriptions of gravity interactions derived in the low energy string theory [1] and/or certain noncommutative [2] and quantum group generalizations [3] of gravity and field theory. Such effective models can be expressed in terms of geometries with torsion and nonmetricity in the framework of metric–affine gravity (MAG) [4] and a subclass of such theories can be expressed as an effective Einstein–Proca gravity derived via irreducible decompositions [5].

In a recent work [6] we developed a unified scheme to the geometry of anholonomic frames with associated nonlinear connection (N–connection) structure for a large number of gauge and gravity models with locally isotropic and anisotropic interactions and nontrivial

torsion and nonmetricity contributions and effective generalized Finsler–Weyl–Riemann–Cartan geometries derived from MAG. The synthesis of metric–affine and Finsler like theories was inspired by a number of exact solutions parametrized by generic off–diagonal metrics and anholonomic frames in Einstein, Einstein–Cartan, gauge and string gravity [7, 8]. The resulting formalism admits inclusion of locally anisotropic spinor interactions and extensions to noncommutative geometry and string/brane gravity [9, 10]. We concluded that the geometry of metric–affine spaces enabled with an additional N–connection structure is sufficient not only to model the bulk of physically important non–Riemannian geometries on (pseudo) Riemannian spaces but also states the conditions when such effective spaces with generic anisotropy can be defined as certain generalized Finsler–affine geometric configurations constructed as exact solutions of field equations. It was elaborated a detailed classification of such spacetimes provided with N–connection structure.

If in the Ref. [6] we paid attention to the geometrical (pre–dynamical) aspects of the generalized Finsler–affine configurations derived in MAG, the aim of this paper (the second partner) is to formulate a variatonal formalism of deriving field equations on metric–affine spaces provided with N–connection structure and to state the main theorems for constructing exact off–diagonal solutions in such generalized non–Riemannian gravity theories. We emphasize that generalized Finsler metrics can be generated in string gravity connected to anholonomic metric–affine configurations. In particular, we investigate how the so–called Obukhov's equivalence theorem [5] should be modified as to include various type of Finsler–Lagrange–Hamilton–Cartan metrics, see Refs. [11, 12, 13]. The results of this paper consist a theoretical background for constructing exact solutions in MAG and string gravity in the third partner paper [14] derived as exact solutions of gravitational and matter field equations parametrized by generic off–diagonal metrics (which can not be diagonalized by local coordinate transforms) and anholonomic frames with associated N–connection structure. Such solutions depending on 3-5 variables (generalizing to MAG the results from [7, 8, 9, 10, 15]) differ substantially from those elaborated in Refs. [16]; they define certain extensions to nontrivial torsion and nonmetricity fields of certain generic off–diagonal metrics in general relativity theory.

The plan of the paper is as follows: In Sec. 2 we outline the necessary results on Finsler–affine geometry. Next, in Sec. 3, we formulate the field equations on metric–affine spaces provided with N–connection structure. We consider Lagrangians and derive geometrically the field equations of Finsler–affine gravity. We prove the main theorems for the Einstein–Proca systems distinguished by N–connection structure and analyze possible string gravity corrections by H–fields from the bosonic string theory. There are defined the restrictions on N–connection structures resulting in Einstein–Cartan and Einstein gravity. Section 4 is devoted to extension of the anholonomic frame method in MAG and string gravity. We formulate and prove the main theorems stating the possibility of constructing exact solutions parametrized by generic off–diagonal metrics, nontrivial torsion and nonmetricity structures and possible sources of matter fields. In Sec. 5 we construct three classes of exact solutions. The first class of solutions is stated for five subclasses of two dimensional generalized Finsler geometries modeled in MAG with possible string corrections. The second class of solutions is for MAG with effective variable and inhomogeneous cosmological constant. The third class of solutions are for the string Finsler–affine gravity (i. e. string gravity containing in certain limits Finsler like metrics) with possible nonlinear

three dimensional solitonic interactions, Proca fields with almost vanishing masses, nontrivial torsions and nonmetricity. In Sec. 6 we present the final remarks. In Appendices A, B and C we give respectively the details on the proof of the Theorem 4.1 (stating the components of the Ricci tensor for generalized Finsler–affine spaces), analyze the reduction of nonlinear solutions from five to four dimensions and present a short characterization of five classes of generalized Finsler–affine spaces.

Our basic notations and conventions are those from Ref. [6] and contain an interference of approaches elaborated in MAG and generalized Finsler geometry. The spacetime is considered to be a manifold V^{n+m} of necessary smoothly class of dimension $n+m$. The Greeck indices $\alpha, \beta, ...$ split into subclasses like $\alpha = (i,a)$, $\beta = (j,b)$ ... where the Latin indices $i,j,k,...$ run values $1,2,...n$ and $a,b,c,...$ run values $n+1, n+2, ..., n+m$. We follow the Penrose convention on abstract indices [17] and use underlined indices like $\underline{\alpha} = (\underline{i}, \underline{a})$, for decompositions with respect to coordinate frames. The symbol " $\doteqdot$" will be used is some formulas will be introduced by definition and the end of proofs will be stated by symbol $\blacksquare$. The notations for connections $\Gamma^{\alpha}{}_{\beta\gamma}$, metrics $g_{\alpha\beta}$ and frames e_α and coframes ϑ^β, or another geometrical and physical objects, are the standard ones from MAG if a nonlinear connection (N–connection) structure is not emphasized on the spacetime. If a N–connection and corresponding anholonomic frame structure are prescribed, we use "boldfaced" symbols with possible splitting of the objects and indices like $\mathbf{V}^{n+m}$, $\mathbf{\Gamma}^{\alpha}{}_{\beta\gamma} = \left(L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc}\right)$, $\mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab})$, $\mathbf{e}_\alpha = (e_i, e_a)$, ...being distinguished by N–connection (in brief, there are used the terms d–objects, d–tensor, d–connection).

2 Metric–Affine and Generalized Finsler Gravity

In this section we recall some basic facts on metric–affine spaces provided with nonlinear connection (N–connection) structure and generalized Finsler–affine geometry [6].

The spacetime is modeled as a manifold V^{n+m} of dimension $n+m$, with $n \geq 2$ and $m \geq 1$, admitting (co) vector/ tangent structures. It is denoted by $\pi^T : TV^{n+m} \to TV^n$ the differential of the map $\pi : V^{n+m} \to V^n$ defined as a fiber–preserving morphism of the tangent bundle (TV^{n+m}, τ_E, V^n) to V^{n+m} and of tangent bundle (TV^n, τ, V^n). We consider also the kernel of the morphism π^T as a vector subbundle of the vector bundle $(TV^{n+m}, \tau_E, V^{n+m})$. The kernel defines the vertical subbundle over V^{n+m}, s denoted as $(vV^{n+m}, \tau_V, V^{n+m})$. We parametrize the local coordinates of a point $u \in V^{n+m}$ as $u^\alpha = (x^i, y^a)$, where the values of indices are $i,j,k,... = 1,2,...,n$ and $a,b,c,... = n+1, n+2, ..., n+m$. The inclusion mapping is written as $i : vV^{n+m} \to TV^{n+m}$.

A nonlinear connection (N–connection) $\mathbf{N}$ in a space (V^{n+m}, π, V^n) is a morphism of manifolds $N : TV^{n+m} \to vV^{n+m}$ defined by the splitting on the left of the exact sequence

$$0 \to vV^{n+m} \to TV^{n+m}/vV^{n+m} \to 0. \tag{1}$$

The kernel of the morphism $\mathbf{N}$ is a subbundle of $(TV^{n+m}, \tau_E, V^{n+m})$, called the horizontal subspace and denoted by $(hV^{n+m}, \tau_H, V^{n+m})$. Every tangent bundle $(TV^{n+m}, \tau_E, V^{n+m})$ provided with a N–connection structure is a Whitney sum of the vertical and horizontal subspaces (in brief, h- and v– subspaces), i. e.

$$TV^{n+m} = hV^{n+m} \oplus vV^{n+m}. \tag{2}$$

We note that the exact sequence (1) defines the N–connection in a global coordinate free form resulting in invariant splitting (2) (see details in Refs. [18, 12] stated for vector and tangent bundles and generalizations on covector bundles, superspaces and noncommutative spaces [13] and [9]).

A N–connection structure prescribes a class of vielbein transforms

$$A_\alpha^{\ \underline{\alpha}}(u) = \mathbf{e}_\alpha^{\ \underline{\alpha}} = \begin{bmatrix} e_i^{\ \underline{i}}(u) & N_i^b(u) e_b^{\ \underline{a}}(u) \\ 0 & e_a^{\ \underline{a}}(u) \end{bmatrix}, \quad (3)$$

$$A^\beta_{\ \underline{\beta}}(u) = \mathbf{e}^\beta_{\ \underline{\beta}} = \begin{bmatrix} e^i_{\ \underline{i}}(u) & -N_k^b(u) e^k_{\ \underline{i}}(u) \\ 0 & e^a_{\ \underline{a}}(u) \end{bmatrix}, \quad (4)$$

in particular case $e_i^{\ \underline{i}} = \delta_i^{\underline{i}}$ and $e_a^{\ \underline{a}} = \delta_a^{\underline{a}}$ with $\delta_i^{\underline{i}}$ and $\delta_a^{\underline{a}}$ being the Kronecker symbols, defining a global splitting of $\mathbf{V}^{n+m}$ into "horizontal" and "vertical" subspaces with the N–vielbein structure

$$\mathbf{e}_\alpha = \mathbf{e}_\alpha^{\ \underline{\alpha}} \partial_{\underline{\alpha}} \text{ and } \vartheta^\beta = \mathbf{e}^\beta_{\ \underline{\beta}} du^{\underline{\beta}}.$$

We adopt the convention that for the spaces provided with N–connection structure the geometrical objects can be denoted by "boldfaced" symbols if it would be necessary to distinguish such objects from similar ones for spaces without N–connection.

A N–connection $\mathbf{N}$ in a space $\mathbf{V}^{n+m}$ is parametrized by its components $N_i^a(u) = N_i^a(x,y)$,

$$\mathbf{N} = N_i^a(u) d^i \otimes \partial_a$$

and characterized by the N–connection curvature

$$\mathbf{\Omega} = \frac{1}{2} \Omega_{ij}^a d^i \wedge d^j \otimes \partial_a,$$

with N–connection curvature coefficients

$$\Omega_{ij}^a = \delta_{[j} N_{i]}^a = \frac{\partial N_i^a}{\partial x^j} - \frac{\partial N_j^a}{\partial x^i} + N_i^b \frac{\partial N_j^a}{\partial y^b} - N_j^b \frac{\partial N_i^a}{\partial y^b}. \quad (5)$$

On spaces provided with N–connection structure, we have to use 'N–elongated' operators like δ_j in (5) instead of usual partial derivatives. They are defined by the vielbein configuration induced by the N–connection, the N–elongated partial derivatives (in brief, N–derivatives)

$$\mathbf{e}_\alpha \doteqdot \delta_\alpha = (\delta_i, \partial_a) \equiv \frac{\delta}{\delta u^\alpha} = \left(\frac{\delta}{\delta x^i} = \partial_i - N_i^a(u) \partial_a, \frac{\partial}{\partial y^a} \right) \quad (6)$$

and the N–elongated differentials (in brief, N–differentials)

$$\vartheta^\beta \doteqdot \delta^{\ \beta} = \left(d^i, \delta^a \right) \equiv \delta u^\alpha = \left(\delta x^i = dx^i, \delta y^a = dy^a + N_i^a(u) dx^i \right) \quad (7)$$

called also, respectively, the N–frame and N–coframe. There are used both type of denotations $\mathbf{e}_\alpha \doteqdot \delta_\alpha$ and $\vartheta^\beta \doteqdot \delta^{\ \alpha}$ in order to preserve a connection to denotations from Refs. [12, 7, 8, 9]. The 'boldfaced' symbols $\mathbf{e}_\alpha$ and ϑ^β will be considered in order to emphasize that they define N–adapted vielbeins but the symbols δ_α and $\delta^{\ \beta}$ will be used for the N–elongated partial derivatives and, respectively, differentials.

The N–coframe (7) satisfies the anholonomy relations

$$[\delta_\alpha, \delta_\beta] = \delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha = \mathbf{w}^{\gamma}{}_{\alpha\beta}(u)\,\delta_\gamma \tag{8}$$

with nontrivial anholonomy coefficients $\mathbf{w}^{\alpha}_{\beta\gamma}(u)$ computed as

$$\mathbf{w}^{a}{}_{ji} = -\mathbf{w}^{a}{}_{ij} = \Omega^{a}_{ij},\ \mathbf{w}^{b}{}_{ia} = -\mathbf{w}^{b}{}_{ai} = \partial_a N^b_i. \tag{9}$$

The distinguished objects (by a N–connection on a spaces $\mathbf{V}^{n+m}$) are introduced in a coordinate free form as geometric objects adapted to the splitting (2). In brief, they are called d–objects, d–tensor, d–connections, d–metrics....

There is an important class of linear connections adapted to the N–connection structure:

A d–connection $\mathbf{D}$ on a space $\mathbf{V}^{n+m}$ is defined as a linear connection D conserving under a parallelism the global decomposition (2).

The N–adapted components $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ of a d-connection $\mathbf{D}_\alpha = (\delta_\alpha \rfloor \mathbf{D})$ are defined by the equations

$$\mathbf{D}_\alpha\delta_\beta = \mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\delta_\gamma,$$

from which one immediately follows

$$\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}(u) = (\mathbf{D}_\alpha\delta_\beta)\rfloor\delta^\gamma. \tag{10}$$

The operations of h- and v-covariant derivations, $D^{[h]}_k = \{L^i_{jk}, L^a_{bk}\}$ and $D^{[v]}_c = \{C^i_{jk}, C^a_{bc}\}$ are introduced as corresponding h- and v–parametrizations of (10),

$$L^i_{jk} = (\mathbf{D}_k\delta_j)\rfloor d^i, \quad L^a_{bk} = (\mathbf{D}_k\partial_b)\rfloor\delta^a,\ C^i_{jc} = (\mathbf{D}_c\delta_j)\rfloor d^i, \quad C^a_{bc} = (\mathbf{D}_c\partial_b)\rfloor\delta^a.$$

The components $\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \left(L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc}\right)$ completely define a d–connection $\mathbf{D}$ in $\mathbf{V}^{n+m}$.

A metric structure $\mathbf{g}$ on a space $\mathbf{V}^{n+m}$ is defined as a symmetric covariant tensor field of type $(0,2)$, $g_{\alpha\beta,}$ being nondegenerate and of constant signature on $\mathbf{V}^{n+m}$. A N–connection $\mathbf{N} = \{N^{\underline{b}}_{\underline{i}}(u)\}$ and a metric structure $\mathbf{g} = g_{\underline{\alpha}\underline{\beta}}du^{\underline{\alpha}} \otimes du^{\underline{\beta}}$ on $\mathbf{V}^{n+m}$ are mutually compatible if there are satisfied the conditions

$$\mathbf{g}(\delta_{\underline{i}}, \partial_{\underline{a}}) = 0,\ \text{or equivalently,}\ g_{\underline{ia}}(u) - N^{\underline{b}}_{\underline{i}}(u)\,h_{\underline{ab}}(u) = 0,$$

where $h_{\underline{ab}} \doteqdot \mathbf{g}(\partial_{\underline{a}}, \partial_{\underline{b}})$ and $g_{\underline{ia}} \doteqdot \mathbf{g}(\partial_{\underline{i}}, \partial_{\underline{a}})$ resulting in

$$N^b_i(u) = h^{ab}(u)\,g_{ia}(u)$$

(the matrix h^{ab} is inverse to h_{ab}; for simplicity, we do not underline the indices in the last formula). In consequence, we define an invariant h–v–decomposition of metric (in brief, a d–metric)

$$\mathbf{g}(X,Y) = h\mathbf{g}(X,Y) + v\mathbf{g}(X,Y).$$

With respect to a N–coframe (7), the d–metric is written

$$\mathbf{g} = \mathbf{g}_{\alpha\beta}(u)\,\delta^\alpha \otimes \delta^\beta = g_{ij}(u)\,d^i \otimes d^j + h_{ab}(u)\,\delta^a \otimes \delta^b, \tag{11}$$

where $g_{ij} \doteqdot \mathbf{g}(\delta_i, \delta_j)$. The d–metric (11) can be equivalently written in "off–diagonal" with respect to a coordinate basis defined by usual local differentials $du^\alpha = (dx^i, dy^a)$,

$$\underline{g}_{\alpha\beta} = \left[\begin{array}{cc} g_{ij} + N_i^a N_j^b h_{ab} & N_j^e h_{ae} \\ N_i^e h_{be} & h_{ab} \end{array} \right]. \tag{12}$$

A metric, for instance, parametrized in the form (12) is generic off–diagonal if it can not be diagonalized by any coordinate transforms. The anholonomy coefficients (9) do not vanish for the off–diagonal form (12) and the equivalent d–metric (11).

The nonmetricity d–field

$$\mathcal{Q} = \mathbf{Q}_{\alpha\beta}\vartheta^\alpha \otimes \vartheta^\beta = \mathbf{Q}_{\alpha\beta}\delta^{\ \alpha} \otimes \delta^\beta$$

on a space $\mathbf{V}^{n+m}$ provided with N–connection structure is defined by a d–tensor field with the coefficients

$$\mathbf{Q}_{\alpha\beta} \doteqdot -\mathbf{D}\mathbf{g}_{\alpha\beta} \tag{13}$$

where the covariant derivative $\mathbf{D}$ is for a d–connection (10) $\mathbf{\Gamma}^\gamma_{\ \alpha} = \mathbf{\Gamma}^\gamma_{\ \alpha\beta}\vartheta^\beta$ with $\mathbf{\Gamma}^\gamma_{\ \alpha\beta} = \left(L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc}\right)$.

A linear connection D_X is compatible with a d–metric $\mathbf{g}$ if

$$D_X \mathbf{g} = 0, \tag{14}$$

i. e. if $Q_{\alpha\beta} \equiv 0$. In a space provided with N–connection structure, the metricity condition (14) may split into a set of compatibility conditions on h- and v– subspaces,

$$D^{[h]}(h\mathbf{g}) = 0, D^{[v]}(h\mathbf{g}) = 0, D^{[h]}(v\mathbf{g}) = 0, D^{[v]}(v\mathbf{g}) = 0. \tag{15}$$

For instance, if $D^{[v]}(h\mathbf{g}) = 0$ and $D^{[h]}(v\mathbf{g}) = 0$, but, in general, $D^{[h]}(h\mathbf{g}) \neq 0$ and $D^{[v]}(v\mathbf{g}) \neq 0$ we have a nontrivial nonmetricity d–field $\mathbf{Q}_{\alpha\beta} = \mathbf{Q}_{\gamma\alpha\beta}\vartheta^\gamma$ with irreducible h–v–components $\mathbf{Q}_{\gamma\alpha\beta} = (Q_{ijk}, Q_{abc})$.

In a metric–affine space, by acting on forms with a covariant derivative D, we can also define another very important geometric objects (the 'gravitational field potentials', the torsion and, respectively, curvature; see [4]):

$$\mathbf{T}^\alpha \doteqdot \mathbf{D}\vartheta^\alpha = \delta\vartheta^\alpha + \mathbf{\Gamma}^\gamma_{\ \beta} \wedge \vartheta^\beta \tag{16}$$

and

$$\mathbf{R}^\alpha_{\ \beta} \doteqdot \mathbf{D}\mathbf{\Gamma}^\alpha_{\ \beta} = \delta\mathbf{\Gamma}^\alpha_{\ \beta} - \mathbf{\Gamma}^\gamma_{\ \beta} \wedge \mathbf{\Gamma}^\alpha_{\ \ \gamma} \tag{17}$$

For spaces provided with N–connection structures, we consider the same formulas but for "boldfaced" symbols and change the usual differential d into N-adapted operator δ.

A general affine (linear) connection $D = \bigtriangledown + Z = \{\Gamma^\gamma_{\ \beta\alpha} = \Gamma^\gamma_{\ \bigtriangledown\beta\alpha} + Z^\gamma_{\ \beta\alpha}\}$

$$\Gamma^\gamma_{\ \alpha} = \Gamma^\gamma_{\ \alpha\beta}\vartheta^\beta, \tag{18}$$

can always be decomposed into the Riemannian $\Gamma^\alpha_{\bigtriangledown\ \beta}$ and post–Riemannian $Z^\alpha_{\ \beta}$ parts [4, 5],

$$\Gamma^\alpha_{\ \beta} = \Gamma^\alpha_{\bigtriangledown\ \beta} + Z^\alpha_{\ \beta}. \tag{19}$$

The distorsion 1-form $Z^{\alpha}_{\ \beta}$ from (19) is expressed in terms of torsion and nonmetricity,

$$Z_{\alpha\beta} = e_\beta \rfloor T_\alpha - e_\alpha \rfloor T_\beta + \frac{1}{2}\left(e_\alpha \rfloor e_\beta \rfloor T_\gamma\right)\vartheta^\gamma + \left(e_\alpha \rfloor Q_{\beta\gamma}\right)\vartheta^\gamma - \left(e_\beta \rfloor Q_{\alpha\gamma}\right)\vartheta^\gamma + \frac{1}{2}Q_{\alpha\beta} \qquad (20)$$

where T_α is defined as (16) and $Q_{\alpha\beta} \doteqdot -Dg_{\alpha\beta}$. (We note that $Z^{\alpha}_{\ \beta}$ are $N_{\alpha\beta}$ from Ref. [5], but in our works we use the symbol N for N–connections .) For $Q_{\beta\gamma} = 0$, we obtain from (20) just the distorsion for the Riemannian–Cartan geometry [19].

By substituting arbitrary (co) frames, metrics and linear connections into N–adapted ones,

$$e_\alpha \rightarrow \mathbf{e}_\alpha, \vartheta^\beta \rightarrow \vartheta^\beta, g_{\alpha\beta} \rightarrow \mathbf{g}_{\alpha\beta} = \left(g_{ij}, h_{ab}\right), \Gamma^{\gamma}_{\ \alpha} \rightarrow \mathbf{\Gamma}^{\gamma}_{\ \alpha},$$

with $\mathbf{Q}_{\alpha\beta} = \mathbf{Q}_{\gamma\alpha\beta}\vartheta^\gamma$ and $\mathbf{T}^\alpha$ as in (16), into respective formulas (18), (19) and (20), we can define an affine connection $\mathbf{D} = \bigtriangledown + \mathbf{Z} = [\mathbf{\Gamma}^{\gamma}_{\ \beta\alpha}]$ with respect to N–adapted (co) frames,

$$\mathbf{\Gamma}^{\gamma}_{\ \ \alpha} = \mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}\vartheta^\beta, \qquad (21)$$

with

$$\mathbf{\Gamma}^{\alpha}_{\ \beta} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta} + \mathbf{Z}^{\alpha}_{\ \ \beta}, \qquad (22)$$

where

$$\mathbf{\Gamma}^{\bigtriangledown}_{\ \gamma\alpha} = \frac{1}{2}\left[\mathbf{e}_\gamma \rfloor\ \delta\vartheta_\alpha - \mathbf{e}_\alpha \rfloor\ \delta\vartheta_\gamma - \left(\mathbf{e}_\gamma \rfloor\ \mathbf{e}_\alpha \rfloor\ \delta\vartheta_\beta\right) \wedge \vartheta^\beta\right], \qquad (23)$$

and

$$\mathbf{Z}_{\alpha\beta} = \mathbf{e}_\beta \rfloor \mathbf{T}_\alpha - \mathbf{e}_\alpha \rfloor \mathbf{T}_\beta + \frac{1}{2}\left(\mathbf{e}_\alpha \rfloor \mathbf{e}_\beta \rfloor \mathbf{T}_\gamma\right)\vartheta^\gamma + \left(\mathbf{e}_\alpha \rfloor \mathbf{Q}_{\beta\gamma}\right)\vartheta^\gamma - \left(\mathbf{e}_\beta \rfloor \mathbf{Q}_{\alpha\gamma}\right)\vartheta^\gamma + \frac{1}{2}\mathbf{Q}_{\alpha\beta}. \qquad (24)$$

The h– and v–components of $\mathbf{\Gamma}^{\alpha}_{\ \beta}$ from (22) consists from the components of $\mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta}$ (considered for (23)) and of $\mathbf{Z}_{\alpha\beta}$ with $\mathbf{Z}^{\alpha}_{\ \ \gamma\beta} = \left(Z^i_{jk}, Z^a_{bk}, Z^i_{jc}, Z^a_{bc}\right)$. We note that for $\mathbf{Q}_{\alpha\beta} = 0$, the distorsion 1–form $\mathbf{Z}_{\alpha\beta}$ defines a Riemann–Cartan geometry adapted to the N–connection structure.

A distinguished metric–affine space $\mathbf{V}^{n+m}$ is defined as a usual metric–affine space additionally enabled with a N–connection structure $\mathbf{N} = \{N^a_i\}$ inducing splitting into respective irreducible horizontal and vertical subspaces of dimensions n and m. This space is provided with independent d–metric (11) and affine d–connection (10) structures adapted to the N–connection.

If a space $\mathbf{V}^{n+m}$ is provided with both N–connection $\mathbf{N}$ and d–metric $\mathbf{g}$ structures, there is a unique linear symmetric and torsionless connection $\bigtriangledown$, called the Levi–Civita connection, being metric compatible such that $\bigtriangledown_\gamma \mathbf{g}_{\alpha\beta} = 0$ for $\mathbf{g}_{\alpha\beta} = \left(g_{ij}, h_{ab}\right)$, see (11), with the coefficients

$$\mathbf{\Gamma}^{\bigtriangledown}_{\alpha\beta\gamma} = \mathbf{g}\left(\delta_\alpha, \bigtriangledown_\gamma \delta_\beta\right) = \mathbf{g}_{\alpha\tau}\mathbf{\Gamma}^{\tau}_{\bigtriangledown\beta\gamma},$$

computed as

$$\mathbf{\Gamma}^{\bigtriangledown}_{\alpha\beta\gamma} = \frac{1}{2}\left[\delta_\beta \mathbf{g}_{\alpha\gamma} + \delta_\gamma \mathbf{g}_{\beta\alpha} - \delta_\alpha \mathbf{g}_{\gamma\beta} + \mathbf{g}_{\alpha\tau}\mathbf{w}^{\tau}_{\gamma\beta} + \mathbf{g}_{\beta\tau}\mathbf{w}^{\tau}_{\alpha\gamma} - \mathbf{g}_{\gamma\tau}\mathbf{w}^{\tau}_{\beta\alpha}\right] \qquad (25)$$

with respect to N–frames $\mathbf{e}_\beta \doteqdot \delta_\beta$ (6) and N–coframes $\vartheta^\alpha \doteqdot \delta^\alpha$ (7).

We note that the Levi–Civita connection is not adapted to the N–connection structure. Se, we can not state its coefficients in an irreducible form for the h– and v–subspaces. There is a type of d–connections which are similar to the Levi–Civita connection but satisfying certain metricity conditions adapted to the N–connection. They are introduced as metric d–connections $\mathbf{D} = \left(D^{[h]}, D^{[v]}\right)$ in a space $\mathbf{V}^{n+m}$ satisfying the metricity conditions if and only if

$$D_k^{[h]} g_{ij} = 0,\ D_a^{[v]} g_{ij} = 0,\ D_k^{[h]} h_{ab} = 0,\ D_a^{[h]} h_{ab} = 0. \tag{26}$$

Let us consider an important example: The canonical d–connection $\widehat{\mathbf{D}} = \left(\widehat{D}^{[h]}, \widehat{D}^{[v]}\right)$, equivalently $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha} = \widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}$, is defined by the h– v–irreducible components $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = (\widehat{L}^{i}_{jk}, \widehat{L}^{a}_{bk}, \widehat{C}^{i}_{jc}, \widehat{C}^{a}_{bc})$, where

$$\begin{aligned}
\widehat{L}^{i}_{jk} &= \frac{1}{2}g^{ir}\left(\frac{\delta g_{jk}}{\delta x^{k}} + \frac{\delta g_{kr}}{\delta x^{j}} - \frac{\delta g_{jk}}{\delta x^{r}}\right), \\
\widehat{L}^{a}_{bk} &= \frac{\partial N^{a}_{k}}{\partial y^{b}} + \frac{1}{2}h^{ac}\left(\frac{\delta h_{bc}}{\delta x^{k}} - \frac{\partial N^{d}_{k}}{\partial y^{b}}h_{dc} - \frac{\partial N^{d}_{k}}{\partial y^{c}}h_{db}\right), \\
\widehat{C}^{i}_{jc} &= \frac{1}{2}g^{ik}\frac{\partial g_{jk}}{\partial y^{c}}, \\
\widehat{C}^{a}_{bc} &= \frac{1}{2}h^{ad}\left(\frac{\partial h_{bd}}{\partial y^{c}} + \frac{\partial h_{cd}}{\partial y^{b}} - \frac{\partial h_{bc}}{\partial y^{d}}\right).
\end{aligned} \tag{27}$$

satisfying the torsionless conditions for the h–subspace and v–subspace, respectively, $\widehat{T}^{i}_{jk} = \widehat{T}^{a}_{bc} = 0$.

The components of the Levi–Civita connection $\mathbf{\Gamma}^{\tau}_{\bigtriangledown\beta\gamma}$ and the irreducible components of the canonical d–connection $\widehat{\mathbf{\Gamma}}^{\tau}{}_{\beta\gamma}$ are related by formulas

$$\mathbf{\Gamma}^{\tau}_{\bigtriangledown\beta\gamma} = \left(\widehat{L}^{i}_{jk}, \widehat{L}^{a}_{bk} - \frac{\partial N^{a}_{k}}{\partial y^{b}}, \widehat{C}^{i}_{jc} + \frac{1}{2}g^{ik}\Omega^{a}_{jk}h_{ca}, \widehat{C}^{a}_{bc}\right), \tag{28}$$

where Ω^{a}_{jk} is the N–connection curvature (5).

We can define and calculate the irreducible components of torsion and curvature in a space $\mathbf{V}^{n+m}$ provided with additional N–connection structure (these could be any metric–affine spaces [4], or their particular, like Riemann–Cartan [19], cases with vanishing non-metricity and/or torsion, or any (co) vector / tangent bundles like in Finsler geometry and generalizations).

The torsion $\mathbf{T}^{\alpha}_{.\beta\gamma} = (T^{i}_{.jk}, T^{i}_{ja}, T^{a}_{.ij}, T^{a}_{.bi}, T^{a}_{.bc})$ of a d–connection $\mathbf{\Gamma}^{\gamma}_{\alpha\beta} = (L^{i}_{jk}, L^{a}_{bk}, C^{i}_{jc}, C^{a}_{bc})$ (10) has irreducible h- v–components (d–torsions)

$$\begin{aligned}
T^{i}_{.jk} &= -T^{i}_{kj} = L^{i}_{jk} - L^{i}_{kj}, \quad T^{i}_{ja} = -T^{i}_{aj} = C^{i}_{.ja},\ T^{a}_{.ji} = -T^{a}_{.ij} = \frac{\delta N^{a}_{i}}{\delta x^{j}} - \frac{\delta N^{a}_{j}}{\delta x^{i}} = \Omega^{a}_{.ji}, \\
T^{a}_{.bi} &= -T^{a}_{.ib} = P^{a}_{.bi} = \frac{\partial N^{a}_{i}}{\partial y^{b}} - L^{a}_{.bj},\ T^{a}_{.bc} = -T^{a}_{.cb} = S^{a}_{.bc} = C^{a}_{bc} - C^{a}_{cb}.
\end{aligned} \tag{29}$$

We note that on (pseudo) Riemanian spacetimes the d–torsions can be induced by the N–connection coefficients and reflect an anholonomic frame structure. Such objects

vanish when we transfer our considerations with respect to holonomic bases for a trivial N–connection and zero "vertical" dimension.

The curvature $\mathbf{R}^{\alpha}_{.\beta\gamma\tau} = (R^{i}{}_{hjk}, R^{a}{}_{bjk}, P^{i}{}_{jka}, P^{c}{}_{bka}, S^{i}{}_{jbc}, S^{a}{}_{bcd})$ of a d–connection $\mathbf{\Gamma}^{\gamma}_{\alpha\beta} = (L^{i}_{jk}, L^{a}_{bk}, C^{i}_{jc}, C^{a}_{bc})$ (10) has irreducible h- v–components (d–curvatures)

$$
\begin{aligned}
R^{i}{}_{hjk} &= \frac{\delta L^{i}_{.hj}}{\delta x^{k}} - \frac{\delta L^{i}_{.hk}}{\delta x^{j}} + L^{m}_{.hj}L^{i}_{mk} - L^{m}_{.hk}L^{i}_{mj} - C^{i}_{.ha}\Omega^{a}_{.jk}, \\
R^{a}{}_{bjk} &= \frac{\delta L^{a}_{.bj}}{\delta x^{k}} - \frac{\delta L^{a}_{.bk}}{\delta x^{j}} + L^{c}_{.bj}L^{a}_{.ck} - L^{c}_{.bk}L^{a}_{.cj} - C^{a}_{.bc}\,\Omega^{c}_{.jk}, \\
P^{i}{}_{jka} &= \frac{\partial L^{i}_{.jk}}{\partial y^{k}} - \left(\frac{\partial C^{i}_{.ja}}{\partial x^{k}} + L^{i}_{.lk}C^{l}_{.ja} - L^{l}_{.jk}C^{i}_{.la} - L^{c}_{.ak}C^{i}_{.jc}\right) + C^{i}_{.jb}P^{b}_{.ka}, \\
P^{c}{}_{bka} &= \frac{\partial L^{c}_{.bk}}{\partial y^{a}} - \left(\frac{\partial C^{c}_{.ba}}{\partial x^{k}} + L^{c}_{.dk}C^{d}_{.ba} - L^{d}_{.bk}C^{c}_{.da} - L^{d}_{.ak}C^{c}_{.bd}\right) + C^{c}_{.bd}P^{d}_{.ka}, \\
S^{i}{}_{jbc} &= \frac{\partial C^{i}_{.jb}}{\partial y^{c}} - \frac{\partial C^{i}_{.jc}}{\partial y^{b}} + C^{h}_{.jb}C^{i}_{.hc} - C^{h}_{.jc}C^{i}_{hb}, \\
S^{a}{}_{bcd} &= \frac{\partial C^{a}_{.bc}}{\partial y^{d}} - \frac{\partial C^{a}_{.bd}}{\partial y^{c}} + C^{e}_{.bc}C^{a}_{.ed} - C^{e}_{.bd}C^{a}_{.ec}.
\end{aligned} \tag{30}
$$

The components of the Ricci tensor

$$\mathbf{R}_{\alpha\beta} = \mathbf{R}^{\tau}{}_{\alpha\beta\tau}$$

with respect to a locally adapted frame (6) has four irreducible h- v–components $\mathbf{R}_{\alpha\beta} = (R_{ij}, R_{ia}, R_{ai}, S_{ab})$, where

$$
\begin{aligned}
R_{ij} &= R^{k}{}_{ijk}, \quad R_{ia} = -\,{}^{2}P_{ia} = -P^{k}{}_{ika}, \\
R_{ai} &= {}^{1}P_{ai} = P^{b}{}_{aib}, \quad S_{ab} = S^{c}{}_{abc}.
\end{aligned} \tag{31}
$$

We point out that because, in general, ${}^{1}P_{ai} \neq {}^{2}P_{ia}$ the Ricci d–tensor is non symmetric.

Having defined a d–metric of type (11) in $\mathbf{V}^{n+m}$, we can introduce the scalar curvature of a d–connection $\mathbf{D}$,

$$\overleftarrow{\mathbf{R}} = \mathbf{g}^{\alpha\beta}\mathbf{R}_{\alpha\beta} = R + S, \tag{32}$$

where $R = g^{ij}R_{ij}$ and $S = h^{ab}S_{ab}$ and define the distinguished form of the Einstein tensor (the Einstein d–tensor),

$$\mathbf{G}_{\alpha\beta} \doteqdot \mathbf{R}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\overleftarrow{\mathbf{R}}. \tag{33}$$

The introduced geometrical objects are extremely useful in definition of field equations of MAG and string gravity with nontrivial N–connection structure.

3 N–Connections and Field Equations

The field equations of metric–affine gravity (in brief, MAG) [4, 5] can be reformulated with respect to frames and coframes consisting from mixed holonomic and anholonomic components defined by the N–connection structure. In this case, various type of (pseudo)

Riemannian, Riemann–Cartan and generalized Finsler metrics and additional torsion and nonmetricity sructures with very general local anisotropy can be embedded into MAG. It is known that in a metric–affine spacetime the curvature, torsion and nonmetricity have correspondingly eleven, three and four irreducible pieces. If the N–connection is defined in a metric–affine spacetime, every irreducible component of curvature splits additionally into six h- and v– components (30), every irreducible component of torsion splits additionally into five h- and v– components (29) and every irreducible component of nonmetricity splits additionally into two h- and v– components (defined by splitting of metrics into block ansatz (11)).

3.1 Lagrangians and field equations for Finsler–affine theories

For an arbitrary d–connection $\mathbf{\Gamma}^{\alpha}{}_{\beta}$ in a metric–affine space $\mathbf{V}^{n+m}$ provided with N–connection structure (for simplicity, we can take $n+m=4$) one holds the respective decompositions for d–torsion and nonmetricity d–field,

$$
\begin{aligned}
{}^{(2)}\mathbf{T}^{\alpha} &\doteqdot \frac{1}{3}\vartheta^{\alpha}\wedge\mathbf{T},\ \text{for } \mathbf{T}\doteqdot \mathbf{e}_{\alpha}\rfloor\mathbf{T}^{\alpha}, \\
{}^{(3)}\mathbf{T}^{\alpha} &\doteqdot \frac{1}{3}\ *\left(\vartheta^{\alpha}\wedge\mathbf{P}\right),\ \text{for } \mathbf{P}\doteqdot *\left(\mathbf{T}^{\alpha}\wedge\vartheta_{\alpha}\right), \\
{}^{(1)}\mathbf{T}^{\alpha} &\doteqdot \mathbf{T}^{\alpha}-^{(2)}\mathbf{T}^{\alpha}-^{(3)}\mathbf{T}^{\alpha}
\end{aligned} \tag{34}
$$

and

$$
\begin{aligned}
{}^{(2)}\mathbf{Q}_{\alpha\beta} &\doteqdot \frac{1}{3}*\left(\vartheta_{\alpha}\wedge\mathbf{S}_{\beta}+\vartheta_{\beta}\wedge\mathbf{S}_{\alpha}\right),\ {}^{(4)}\mathbf{Q}_{\alpha\beta}\doteqdot \mathbf{g}_{\alpha\beta}\mathbf{Q}, \\
{}^{(3)}\mathbf{Q}_{\alpha\beta} &\doteqdot \frac{2}{9}\left[\left(\vartheta_{\alpha}\mathbf{e}_{\beta}+\vartheta_{\beta}\mathbf{e}_{\alpha}\right)\rfloor\mathbf{\Lambda}-\frac{1}{2}\mathbf{g}_{\alpha\beta}\mathbf{\Lambda}\right], \\
{}^{(1)}\mathbf{Q}_{\alpha\beta} &\doteqdot \mathbf{Q}_{\alpha\beta}-\ {}^{(2)}\mathbf{Q}_{\alpha\beta}-\ {}^{(3)}\mathbf{Q}_{\alpha\beta}-\ {}^{(4)}\mathbf{Q}_{\alpha\beta},
\end{aligned} \tag{35}
$$

where

$$
\begin{aligned}
\mathbf{Q} &\doteqdot \frac{1}{4}\mathbf{g}^{\alpha\beta}\mathbf{Q}_{\alpha\beta},\ \mathbf{\Lambda}\doteqdot \vartheta^{\alpha}\mathbf{e}^{\beta}\rfloor\left(\mathbf{Q}_{\alpha\beta}-\mathbf{Q}\mathbf{g}_{\alpha\beta}\right), \\
\mathbf{\Theta}_{\alpha} &\doteqdot *\left[\left(\mathbf{Q}_{\alpha\beta}-\mathbf{Q}\mathbf{g}_{\alpha\beta}\right)\wedge\vartheta^{\beta}\right], \\
\mathbf{S}_{\alpha} &\doteqdot \mathbf{\Theta}_{\alpha}-\frac{1}{3}\mathbf{e}_{\alpha}\rfloor\left(\vartheta^{\beta}\wedge\mathbf{\Theta}_{\beta}\right)
\end{aligned}
$$

and the Hodge dual "$*$" is such that $\eta\doteqdot *1$ is the volume 4–form and

$$
\eta_{\alpha}\doteqdot \mathbf{e}_{\alpha}\rfloor\eta=*\vartheta_{\alpha},\ \eta_{\alpha\beta}\doteqdot \mathbf{e}_{\alpha}\rfloor\eta_{\beta}=*\left(\vartheta_{\alpha}\wedge\vartheta_{\beta}\right),\ \eta_{\alpha\beta\gamma}\doteqdot \mathbf{e}_{\gamma}\rfloor\eta_{\alpha\beta},\ \eta_{\alpha\beta\gamma\tau}\doteqdot \mathbf{e}_{\tau}\rfloor\eta_{\alpha\beta\gamma}
$$

with $\eta_{\alpha\beta\gamma\tau}$ being totally antisymmetric. In higher dimensions, we have to consider $\eta\doteqdot *1$ as the volume $(n+m)$–form. For N–adapted h- and v–constructions, we have to consider couples of 'volume' forms $\eta\doteqdot\left(\eta^{[g]}=*^{[g]}1,\eta^{[h]}=*^{[h]}1\right)$ defined correspondingly by $\mathbf{g}_{\alpha\beta}=\left(g_{ij},h_{ab}\right)$.

With respect to N–adapted (co) frames $\mathbf{e}_{\beta}=(\delta_i,\partial_a)$ (6) and $\vartheta^{\alpha}=(d^i,\delta^a)$ (7), the irreducible decompositions (34) split into h- and v–components ${}^{(A)}\mathbf{T}^{\alpha}=\left({}^{(A)}\mathbf{T}^{i},\ {}^{(A)}\mathbf{T}^{a}\right)$

for every $A = 1,2,3,4$. Because, by definition, $\mathbf{Q}_{\alpha\beta} \doteqdot \mathbf{D}\mathbf{g}_{\alpha\beta}$ and $\mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab})$ is a d–metric field, we conclude that in a similar form can be decomposed the nonmetricity, $\mathbf{Q}_{\alpha\beta} = (Q_{ij}, Q_{ab})$. The symmetrizations in formulas (35) hide splittings for ${}^{(1)}\mathbf{Q}_{\alpha\beta}, {}^{(2)}\mathbf{Q}_{\alpha\beta}$ and ${}^{(3)}\mathbf{Q}_{\alpha\beta}$. Nevertheless, the h– and v– decompositions can be derived separately on h– and v–subspaces by distinguishing the interior product $\rfloor = \left(\rfloor^{[h]}, \rfloor^{[v]}\right)$ as to have $\eta_\alpha = (\eta_i = \delta_i \rfloor \eta,\ \eta_a = \partial_a \rfloor \eta)$...and all formulas after decompositions with respect to N–adapted frames (co resulting into a separate relations in h– and v–subspaces, when ${}^{(A)}\mathbf{Q}_{\alpha\beta} = \left({}^{(A)}\mathbf{Q}_{ij},\ {}^{(A)}\mathbf{Q}_{ab}\right)$ for every $A = 1,2,3,4$.

A generalized Finsler–affine theory is described by a Lagrangian

$$\mathcal{L} = \mathcal{L}_{GFA} + \mathcal{L}_{mat},$$

where $\mathcal{L}_{mat}$ represents the Lagrangian of matter fields and

$$\begin{aligned}
\mathcal{L}_{GFA} = & \frac{1}{2\kappa}[-a_{0[Rh]}\mathbf{R}^{ij} \wedge \eta_{ij} - a_{0[Rv]}\mathbf{R}^{ab} \wedge \eta_{ab} - a_{0[Ph]}\mathbf{P}^{ij} \wedge \eta_{ij} - a_{0[Pv]}\mathbf{P}^{ab} \wedge \eta_{ab} \\
& -a_{0[Sh]}\mathbf{S}^{ij} \wedge \eta_{ij} - a_{0[Sv]}\mathbf{S}^{ab} \wedge \eta_{ab} - 2\lambda_{[h]}\eta_{[h]} - 2\lambda_{[v]}\eta_{[v]} \qquad (36) \\
& +\mathbf{T}^i \wedge *^{[h]} \left(\sum_{[A]=1}^{3} a_{[hA]}\ {}^{[A]}\mathbf{T}_i \right) + \mathbf{T}^a \wedge *^{[v]} \left(\sum_{[A]=1}^{3} a_{[vA]}\ {}^{[A]}\mathbf{T}_a \right) \\
& +2\left(\sum_{[I]=2}^{4} c_{[hI]}\ {}^{[I]}\mathbf{Q}_{ij} \right) \wedge \vartheta^i \wedge *^{[h]}\ \mathbf{T}^j + 2\left(\sum_{[I]=2}^{4} c_{[vI]}\ {}^{[I]}\mathbf{Q}_{ab} \right) \wedge \vartheta^a \wedge *^{[v]}\mathbf{T}^b \\
& +\mathbf{Q}_{ij} \wedge \left(\sum_{[I]=1}^{4} b_{[hI]}\ {}^{[I]}\mathbf{Q}^{ij} \right) + \mathbf{Q}_{ab} \wedge \left(\sum_{[I]=1}^{4} b_{[vI]}\ {}^{[I]}\mathbf{Q}^{ab} \right) \\
& +b_{[h5]}\left({}^{[3]}\mathbf{Q}_{ij} \wedge \vartheta^i\right) \wedge *^{[h]}\left({}^{[4]}\mathbf{Q}^{kj} \wedge \vartheta_k\right) + b_{[v5]}\left({}^{[3]}\mathbf{Q}_{ij} \wedge \vartheta^i\right) \wedge *^{[v]}\left({}^{[4]}\mathbf{Q}^{kj} \wedge \vartheta_k\right)]
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2\rho_{[Rh]}}\mathbf{R}^{ij} \wedge *^{[h]}\{\sum_{[I]=1}^{6} w_{[RhI]}\ ({}^{[I]}\mathbf{R}_{ij} - {}^{[I]}\mathbf{R}_{ji}) + w_{[Rh7]}\vartheta_i \wedge [\mathbf{e}_k\rfloor^{[h]}\ (\ {}^{[5]}\mathbf{R}^k{}_j - {}^{[5]}\mathbf{R}_j{}^k)] \\
& + \sum_{[I]=1}^{5} z_{[RhI]}\ ({}^{[I]}\mathbf{R}_{ij} + {}^{[I]}\mathbf{R}_{ji}) + z_{[Rh6]}\vartheta_k \wedge [\mathbf{e}_i\rfloor^{[h]}\ (\ {}^{[2]}\mathbf{R}^k{}_j - {}^{[2]}\mathbf{R}_j{}^k)] \\
& + \sum_{[I]=7}^{9} z_{[RhI]}\vartheta_i \wedge [\mathbf{e}_k\rfloor^{[h]}\ (\ {}^{[I-4]}\mathbf{R}^k{}_j - {}^{[I-4]}\mathbf{R}_j{}^k)]\}
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2\rho_{[Rv]}}\mathbf{R}^{ab} \wedge *^{[v]}\{\sum_{[I]=1}^{6} w_{[RvI]}\ ({}^{[I]}\mathbf{R}_{ab} - {}^{[I]}\mathbf{R}_{ba}) + w_{[Rv7]}\vartheta_a \wedge [\mathbf{e}_c\rfloor^{[v]}\ (\ {}^{[5]}\mathbf{R}^a{}_b - {}^{[5]}\mathbf{R}_b{}^a)] \\
& + \sum_{[I]=1}^{5} z_{[RvI]}\ ({}^{[I]}\mathbf{R}_{ab} + {}^{[I]}\mathbf{R}_{ba}) + z_{[Rv6]}\vartheta_c \wedge [\mathbf{e}_a\rfloor^{[v]}\ (\ {}^{[2]}\mathbf{R}^c{}_b - {}^{[2]}\mathbf{R}_b{}^c)] \\
& + \sum_{[I]=7}^{9} z_{[RvI]}\vartheta_a \wedge [\mathbf{e}_c\rfloor^{[v]}\ (\ {}^{[I-4]}\mathbf{R}^c{}_b - {}^{[I-4]}\mathbf{R}_b{}^c)]\}
\end{aligned}$$

$$-\frac{1}{2\rho_{[Ph]}}\mathbf{P}^{ij}\wedge *^{[h]}\{\sum_{[I]=1}^{6} w_{[PhI]}\ ({}^{[I]}\mathbf{P}_{ij}-\ {}^{[I]}\mathbf{P}_{ji})+w_{[Ph7]}\vartheta_i\wedge[\mathbf{e}_k\rfloor^{[h]}\ (\ {}^{[5]}\mathbf{P}^k{}_j-\ {}^{[5]}\mathbf{P}_j{}^k)]$$

$$+\sum_{[I]=1}^{5} z_{[PhI]}\ ({}^{[I]}\mathbf{P}_{ij}+\ {}^{[I]}\mathbf{P}_{ji})+z_{[Ph6]}\vartheta_k\wedge[\mathbf{e}_i\rfloor^{[h]}\ (\ {}^{[2]}\mathbf{P}^k{}_j-\ {}^{[2]}\mathbf{P}_j{}^k)]$$

$$+\sum_{[I]=7}^{9} z_{[PhI]}\vartheta_i\wedge[\mathbf{e}_k\rfloor^{[h]}\ (\ {}^{[I-4]}\mathbf{P}^k{}_j-\ {}^{[I-4]}\mathbf{P}_j{}^k)]\}-$$

$$\frac{1}{2\rho_{[Pv]}}\mathbf{P}^{ab}\wedge *^{[v]}\{\sum_{[I]=1}^{6} w_{[PvI]}\ ({}^{[I]}\mathbf{P}_{ab}-\ {}^{[I]}\mathbf{P}_{ba})+w_{[Pv7]}\vartheta_a\wedge[\mathbf{e}_c\rfloor^{[v]}\ (\ {}^{[5]}\mathbf{P}^a{}_b-\ {}^{[5]}\mathbf{P}_b{}^a)]$$

$$+\sum_{[I]=1}^{5} z_{[PvI]}\ ({}^{[I]}\mathbf{P}_{ab}+\ {}^{[I]}\mathbf{P}_{ba})+z_{[Pv6]}\vartheta_c\wedge[\mathbf{e}_a\rfloor^{[v]}\ (\ {}^{[2]}\mathbf{P}^c{}_b-\ {}^{[2]}\mathbf{P}_b{}^c)]$$

$$+\sum_{[I]=7}^{9} z_{[PvI]}\vartheta_a\wedge[\mathbf{e}_c\rfloor^{[v]}\ (\ {}^{[I-4]}\mathbf{P}^c{}_b-\ {}^{[I-4]}\mathbf{P}_b{}^c)]\}$$

$$-\frac{1}{2\rho_{[Sh]}}\mathbf{S}^{ij}\wedge *^{[h]}\{\sum_{[I]=1}^{6} w_{[ShI]}\ ({}^{[I]}\mathbf{S}_{ij}-\ {}^{[I]}\mathbf{S}_{ji})+w_{[Sh7]}\vartheta_i\wedge[\mathbf{e}_k\rfloor^{[h]}\ (\ {}^{[5]}\mathbf{S}^k{}_j-\ {}^{[5]}\mathbf{S}_j{}^k)]$$

$$+\sum_{[I]=1}^{5} z_{[ShI]}\ ({}^{[I]}\mathbf{S}_{ij}+\ {}^{[I]}\mathbf{S}_{ji})+z_{[Sh6]}\vartheta_k\wedge[\mathbf{e}_i\rfloor^{[h]}\ (\ {}^{[2]}\mathbf{S}^k{}_j-\ {}^{[2]}\mathbf{S}_j{}^k)]$$

$$+\sum_{[I]=7}^{9} z_{[ShI]}\vartheta_i\wedge[\mathbf{e}_k\rfloor^{[h]}\ (\ {}^{[I-4]}\mathbf{S}^k{}_j-\ {}^{[I-4]}\mathbf{S}_j{}^k)]\}-$$

$$\frac{1}{2\rho_{[Sv]}}\mathbf{S}^{ab}\wedge *^{[v]}\{\sum_{[I]=1}^{6} w_{[SvI]}\ ({}^{[I]}\mathbf{S}_{ab}-\ {}^{[I]}\mathbf{S}_{ba})+w_{[Sv7]}\vartheta_a\wedge[\mathbf{e}_c\rfloor^{[v]}\ (\ {}^{[5]}\mathbf{S}^a{}_b-\ {}^{[5]}\mathbf{S}_b{}^a)]$$

$$+\sum_{[I]=1}^{5} z_{[SvI]}\ ({}^{[I]}\mathbf{S}_{ab}+\ {}^{[I]}\mathbf{S}_{ba})+z_{[Sv6]}\vartheta_c\wedge[\mathbf{e}_a\rfloor^{[v]}\ (\ {}^{[2]}\mathbf{S}^c{}_b-\ {}^{[2]}\mathbf{S}_b{}^c)]$$

$$+\sum_{[I]=7}^{9} z_{[SvI]}\vartheta_a\wedge[\mathbf{e}_c\rfloor^{[v]}\ (\ {}^{[I-4]}\mathbf{S}^c{}_b-\ {}^{[I-4]}\mathbf{S}_b{}^c)]\}.$$

Let us explain the denotations used in (36): The signature is adapted in the form $(-+++)$ and there are considered two Hodge duals, $*^{[h]}$for h–subspace and $*^{[v]}$for v–subspace, and respectively two cosmological constants, $\lambda_{[h]}$ and $\lambda_{[v]}$. The strong gravity coupling constants $\rho_{[Rh]},\rho_{[Rv]},\rho_{[Ph]},....,$ the constants $a_{0[Rh]},a_{0[Rv]},a_{0[Ph]},...,a_{[hA]},a_{[vA]},...\ c_{[hI]},c_{[vI]},...$ are dimensionless and provided with labels $[R],[P],[h],[v]$, emphasizing that the constants are

related, for instance, to respective invariants of curvature, torsion, nonmetricity and their h- and v–decompositions.

The action (36) describes all possible models of Einstein, Einstein–Cartan and all type of Finsler–Lagrange–Cartan–Hamilton gravities which can be modeled on metric affine spaces provided with N–connection structure (i. e. with generic off–diagonal metrics) and derived from quadratic MAG–type Lagrangians.

We can reduce the number of constants in $\mathcal{L}_{GFA} \rightarrow \mathcal{L}'_{GFA}$ if we select the limit resulting in the usual quadratic MAG–Lagrangian [4] for trivial N–connection structure. In this case, all constants for h– and v– decompositions coincide with those from MAG without N–connection structure, for instance,

$$a_0 = a_{0[Rh]} = a_{0[Rv]} = a_{0[Ph]} = ...,\ a_{[A]} = a_{[hA]} = a_{[vA]} = ..., ..., c_{[I]} = c_{[hI]} = c_{[vI]}, ...$$

The Lagrangian (36) can be reduced to a more simple one written in terms of boldfaced symbols (emphasizing a nontrivial N–connection structure) provided with Greek indices,

$$\begin{aligned}
\mathcal{L}'_{GFA} &= \frac{1}{2\kappa}[-a_0 \mathbf{R}^{\alpha\beta} \wedge \eta_{\alpha\beta} - 2\lambda\eta + \mathbf{T}^i \wedge * \left(\sum_{[A]=1}^{3} a_{[A]}\ {}^{[A]}\mathbf{T}_i \right) \\
&+2 \left(\sum_{[I]=2}^{4} c_{[I]}\ {}^{[I]}\mathbf{Q}_{\alpha\beta} \right) \wedge \vartheta^\alpha \wedge *\ \mathbf{T}^\beta + \mathbf{Q}_{\alpha\beta} \wedge \left(\sum_{[I]=1}^{4} b_{[I]}\ {}^{[I]}\mathbf{Q}^{\alpha\beta} \right) \\
&+\mathbf{Q}_{\alpha\beta} \wedge \left(\sum_{[I]=1}^{4} b_{[I]}\ {}^{[I]}\mathbf{Q}^{\alpha\beta} \right) + b_{[5]} \left({}^{[3]}\mathbf{Q}_{\alpha\beta} \wedge \vartheta^\alpha \right) \wedge * \left({}^{[4]}\mathbf{Q}^{\gamma\beta} \wedge \vartheta_\gamma \right)] \\
&-\frac{1}{2\rho}\mathbf{R}^{\alpha\beta} \wedge *[\sum_{[I]=1}^{6} w_{[I]}\ \ {}^{[I]}\mathbf{W}_{\alpha\beta} + w_{[7]}\vartheta_\alpha \wedge \left(\mathbf{e}_\gamma \rfloor\ \ {}^{[5]}\mathbf{W}^{\gamma}{}_{\beta} \right) \\
&+\sum_{[I]=1}^{5} z_{[I]}\ {}^{[I]}\mathbf{Y}_{\alpha\beta} + z_{[6]}\vartheta_\gamma \wedge \left(\mathbf{e}_\alpha \rfloor\ {}^{[2]}\mathbf{Y}^{\gamma}{}_{\beta} \right) + \sum_{[I]=7}^{9} z_{[I]}\vartheta_\alpha \wedge \left(\mathbf{e}_\gamma \rfloor\ \ {}^{[I-4]}\mathbf{Y}^{\gamma}{}_{\beta} \right)].
\end{aligned} \qquad (37)$$

where ${}^{[I]}\mathbf{W}_{\alpha\beta} = {}^{[I]}\mathbf{R}_{\alpha\beta} - {}^{[I]}\mathbf{R}_{\beta\alpha}$ and ${}^{[I]}\mathbf{Y}_{\alpha\beta} = {}^{[I]}\mathbf{R}_{\alpha\beta} + {}^{[I]}\mathbf{R}_{\beta\alpha}$. This action is just for the MAG quadratic theory but with $\mathbf{e}_\alpha$ and ϑ^β being adapted to the N–connection structure as in (6) and (7) with a corresponding splitting of geometrical objects.

The field equations of a metric–affine space provided with N–connection structure, $\mathbf{V}^{n+m} = \left[N_i^a, \mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab}), \mathbf{\Gamma}^{\gamma}_{\ \alpha\beta} = \left(L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc} \right) \right]$, can be obtained by the Noether procedure in its turn being N–adapted to (co) frames $\mathbf{e}_\alpha$ and ϑ^β. At the first step, we parametrize the generalized Finsler–affine Lagrangian and matter Lagrangian respectively as

$$\mathcal{L}'_{GFA} = \mathcal{L}_{[fa]} \left(N_i^a, \mathbf{g}_{\alpha\beta}, \vartheta^\gamma, \mathbf{Q}_{\alpha\beta}, \mathbf{T}^\alpha,\ \mathbf{R}^{\alpha}{}_{\beta} \right)$$

and

$$\mathcal{L}_{mat} = \mathcal{L}_{[m]} \left(N_i^a, \mathbf{g}_{\alpha\beta}, \vartheta^\gamma, \mathbf{\Psi}, \mathbf{D\Psi} \right),$$

where $\mathbf{T}^{\alpha}$ and $\mathbf{R}^{\alpha}{}_{\beta}$ are the curvature of arbitrary d–connection $\mathbf{D}$ and $\mathbf{\Psi}$ represents the matter fields as a p–form. The action $\mathcal{S}$ on $\mathbf{V}^{n+m}$ is written

$$\mathcal{S} = \int \delta^{n+m}u\sqrt{|\mathbf{g}_{\alpha\beta}|}\left[\mathcal{L}_{[fa]} + \mathcal{L}_{[m]}\right] \tag{38}$$

which results in the matter and gravitational (generalized Finsler–affine type) field equations.

Theorem 3.1. *The Yang–Mills type field equations of the generalized Finsler–affine gravity with matter derived by a variational procedure adapted to the N–connection structure are defined by the system*

$$\begin{aligned}
\mathbf{D}\left(\frac{\partial\mathcal{L}_{[m]}}{\partial\left(\mathbf{D}\mathbf{\Psi}\right)}\right) - (-1)^p\,\frac{\partial\mathcal{L}_{[m]}}{\partial\mathbf{\Psi}} &= 0, \\
\mathbf{D}\left(\frac{\partial\mathcal{L}_{[fa]}}{\partial\mathbf{Q}_{\alpha\beta}}\right) + 2\frac{\partial\mathcal{L}_{[fa]}}{\partial\mathbf{g}_{\alpha\beta}} &= -\sigma^{\alpha\beta}, \\
\mathbf{D}\left(\frac{\partial\mathcal{L}_{[fa]}}{\partial\mathbf{T}^{\alpha}}\right) + 2\frac{\partial\mathcal{L}_{[fa]}}{\partial\vartheta^{\alpha}} &= -\mathbf{\Sigma}_{\alpha}, \\
\mathbf{D}\left(\frac{\partial\mathcal{L}_{[fa]}}{\partial\mathbf{R}^{\alpha}{}_{\beta}}\right) + \vartheta^{\beta}\wedge\frac{\partial\mathcal{L}_{[fa]}}{\partial\mathbf{T}^{\alpha}} &= -\mathbf{\Delta}_{\alpha}{}^{\beta},
\end{aligned} \tag{39}$$

where the material currents are defined

$$\sigma^{\alpha\beta} \doteqdot 2\frac{\delta\mathcal{L}_{[m]}}{\delta\mathbf{g}_{\alpha\beta}},\ \ \mathbf{\Sigma}_{\alpha} \doteqdot \frac{\delta\mathcal{L}_{[m]}}{\delta\vartheta^{\alpha}},\ \ \mathbf{\Delta}_{\alpha}{}^{\beta} = \frac{\delta\mathcal{L}_{[m]}}{\delta\mathbf{\Gamma}^{\alpha}{}_{\beta}}$$

for variations "boldfaced" $\delta\mathcal{L}_{[m]}/\delta$ computed with respect to N–adapted (co) frames.

The proof of this theorem consists from N–adapted variational calculus. The equations (39) transforms correspondingly into "MATTER, ZEROTH, FIRST, SECOND" equations of MAG [4] for trivial N–connection structures.

Corollary 3.1. *The system (39) has respectively the h– and v–irreducible components*

$$D^{[h]}\left(\frac{\partial\mathcal{L}_{[m]}}{\partial\left(D^{[h]}\mathbf{\Psi}\right)}\right) + D^{[v]}\left(\frac{\partial\mathcal{L}_{[m]}}{\partial\left(D^{[v]}\mathbf{\Psi}\right)}\right) - (-1)^p\,\frac{\partial\mathcal{L}_{[m]}}{\partial\mathbf{\Psi}} = 0,$$

$$\begin{aligned}
D^{[h]}\left(\frac{\partial\mathcal{L}_{[fa]}}{\partial Q_{ij}}\right) + D^{[v]}\left(\frac{\partial\mathcal{L}_{[fa]}}{\partial Q_{ij}}\right) + 2\frac{\partial\mathcal{L}_{[fa]}}{\partial g_{ij}} &= -\sigma^{ij}, \\
D^{[h]}\left(\frac{\partial\mathcal{L}_{[fa]}}{\partial Q_{ab}}\right) + D^{[v]}\left(\frac{\partial\mathcal{L}_{[fa]}}{\partial Q_{ab}}\right) + 2\frac{\partial\mathcal{L}_{[fa]}}{\partial g_{ab}} &= -\sigma^{ab},
\end{aligned} \tag{40}$$

$$\begin{aligned}
D^{[h]}\left(\frac{\partial\mathcal{L}_{[fa]}}{\partial T^{i}}\right) + D^{[v]}\left(\frac{\partial\mathcal{L}_{[fa]}}{\partial T^{i}}\right) + 2\frac{\partial\mathcal{L}_{[fa]}}{\partial\vartheta^{i}} &= -\Sigma_{i}, \\
D^{[h]}\left(\frac{\partial\mathcal{L}_{[fa]}}{\partial T^{a}}\right) + D^{[v]}\left(\frac{\partial\mathcal{L}_{[fa]}}{\partial T^{a}}\right) + 2\frac{\partial\mathcal{L}_{[fa]}}{\partial\vartheta^{a}} &= -\Sigma_{a},
\end{aligned}$$

$$
\begin{aligned}
D^{[h]}\left(\frac{\partial \mathcal{L}_{[fa]}}{\partial R^{i}{}_{j}}\right) + D^{[v]}\left(\frac{\partial \mathcal{L}_{[fa]}}{\partial R^{i}{}_{j}}\right) + \vartheta^{j} \wedge \frac{\partial \mathcal{L}_{[fa]}}{\partial T^{i}} &= -\mathbf{\Delta}_{i}{}^{j}, \\
D^{[h]}\left(\frac{\partial \mathcal{L}_{[fa]}}{\partial R^{a}{}_{b}}\right) + D^{[v]}\left(\frac{\partial \mathcal{L}_{[fa]}}{\partial R^{a}{}_{b}}\right) + \vartheta^{b} \wedge \frac{\partial \mathcal{L}_{[fa]}}{\partial T^{a}} &= -\mathbf{\Delta}_{a}{}^{b},
\end{aligned}
$$

where

$$
\begin{aligned}
\sigma^{\alpha\beta} &= \left(\sigma^{ij}, \sigma^{ab}\right) \text{ for } \sigma^{ij} \doteqdot 2\frac{\delta \mathcal{L}_{[m]}}{\delta g_{ij}},\ \sigma^{ab} \doteqdot 2\frac{\delta \mathcal{L}_{[m]}}{\delta h_{ab}}, \\
\mathbf{\Sigma}_{\alpha} &= \left(\Sigma_{i}, \Sigma_{a}\right) \text{ for } \Sigma_{i} \doteqdot \frac{\delta \mathcal{L}_{[m]}}{\delta \vartheta^{i}},\ \Sigma_{a} \doteqdot \frac{\delta \mathcal{L}_{[m]}}{\delta \vartheta^{a}}, \\
\mathbf{\Delta}_{\alpha}{}^{\beta} &= \left(\Delta_{i}{}^{j}, \Delta_{a}{}^{b}\right) \text{ for } \Delta_{i}{}^{j} = \frac{\delta \mathcal{L}_{[m]}}{\delta \Gamma^{i}{}_{j}},\ \Delta_{a}{}^{b} = \frac{\delta \mathcal{L}_{[m]}}{\delta \Gamma^{a}{}_{b}}\ .
\end{aligned}
$$

It should be noted that the complete h– v–decomposition of the system (40) can be obtained if we represent the d–connection and curvature forms as

$$
\Gamma^{i}{}_{j} = L^{i}{}_{jk} dx^{j} + C^{i}{}_{ja} \delta y^{a} \text{ and } \Gamma^{a}{}_{b} = L^{a}{}_{bk} dx^{k} + C^{a}{}_{bc} \delta y^{c},
$$

see the d–connection components (10) and

$$
\begin{aligned}
2R^{i}{}_{j} &= R^{i}{}_{jkl} dx^{k} \wedge dx^{l} + P^{i}{}_{jka} dx^{k} \wedge \delta y^{a} + S^{i}{}_{jba} \delta y^{b} \wedge \delta y^{a}, \\
2R^{e}{}_{f} &= R^{e}{}_{fkl} dx^{k} \wedge dx^{l} + P^{e}{}_{fka} dx^{k} \wedge \delta y^{a} + S^{e}{}_{fba} \delta y^{a} \wedge \delta y^{a},
\end{aligned}
$$

see the d–curvature components (30).

Remark 3.1. *For instance, a Finsler configuration can be modeled on a metric affine space provided with N–connection structure,* $\mathbf{V}^{n+m} = \left[N_{i}^{a}, \mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab}),\ {}^{[F]}\widehat{\mathbf{\Gamma}}_{\alpha\beta}^{\gamma}\right]$, *if* $n = m$, *the ansatz for N–connection is of Cartan–Finsler type*

$$
N_{j}^{a} \rightarrow {}^{[F]}N_{j}^{i} = \frac{1}{8}\frac{\partial}{\partial y^{j}}\left[y^{l} y^{k} g_{[F]}^{ih}\left(\frac{\partial g_{hk}^{[F]}}{\partial x^{l}} + \frac{\partial g_{lh}^{[F]}}{\partial x^{k}} - \frac{\partial g_{lk}^{[F]}}{\partial x^{h}}\right)\right],
$$

the d–metric $\mathbf{g}_{\alpha\beta} = \mathbf{g}_{\alpha\beta}^{[F]}$ *is defined by (11) with*

$$
g_{ij}^{[F]} = g_{ij} = h_{ij} = \frac{1}{2}\partial^{2} F / \partial y^{i} \partial y^{j}
$$

and ${}^{[F]}\widehat{\mathbf{\Gamma}}_{\alpha\beta}^{\gamma}$ *is the Finsler canonical d–connection computed as (27). The data should define an exact solution of the system of field equation (40) (equivalently of (39)).*

Similar Remarks hold true for all types of generalized Finsler–affine spaces considered in Tables 1–11 from Ref. [6]. We shall analyze the possibility of modeling various type of locally anisotropic geometries by the Einstein–Proca systems and in string gravity in next subsection.

3.2 Effective Einstein–Proca systems and N–connections

Any affine connection can always be decomposed into (pseudo) Riemannian, $\Gamma^{\alpha}_{\bigtriangledown\ \beta}$, and post–Riemannian, $Z^{\alpha}{}_{\beta}$, parts as $\Gamma^{\alpha}{}_{\beta}=\Gamma^{\alpha}_{\bigtriangledown\ \beta}+Z^{\alpha}{}_{\beta}$, see formulas (19) and (20) (or (22) and (24) if any N–connection structure is prescribed). This mean that it is possible to split all quantities of a metric–affine theory into (pseudo) Riemannian and post–Riemannian pieces, for instance,

$$R^{\alpha}{}_{\beta}=R^{\alpha}_{\bigtriangledown\ \beta}+\bigtriangledown Z^{\alpha}{}_{\beta}+Z^{\alpha}{}_{\gamma}\wedge Z^{\gamma}{}_{\beta}. \tag{41}$$

Under certain assumptions one holds the Obukhov's equivalence theorem according to which the field vacuum metric–affine gravity equations are equivalent to Einstein's equations with an energy–momentum tensor determined by a Proca field [5, 20]. We can generalize the constructions and reformulate the equivalence theorem for generalized Finsler–affine spaces and effective spaces provided with N–connection structure.

Theorem 3.2. *The system of effective field equations of MAG on spaces provided with N–connection structure (39) (equivalently, (40)) for certain ansatz for torsion and non-metricity fields (see (34) and (35))*

$$\begin{aligned} {}^{(1)}\mathbf{T}^{\alpha} &= {}^{(2)}\mathbf{T}^{\alpha}=0,\ \ {}^{(1)}\mathbf{Q}_{\alpha\beta}={}^{(2)}\mathbf{Q}_{\alpha\beta}=0, \\ \mathbf{Q} &= k_0\phi,\ \ \mathbf{\Lambda}=k_1\phi,\ \ \mathbf{T}=k_2\phi, \end{aligned} \tag{42}$$

where $k_0,k_1,k_2=const$ and the Proca 1–form is $\phi=\phi_{\alpha}\vartheta^{\alpha}=\phi_i dx^i+\phi_a\delta y^a$, reduces to the Einstein–Proca system of equations for the canonical d–connection $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$ (27) and massive d–field ϕ_{α},

$$\begin{aligned} \frac{a_0}{2}\eta_{\alpha\beta\gamma}\wedge\widehat{\mathbf{R}}^{\beta\gamma} &= k\ \mathbf{\Sigma}_{\alpha}, \\ \delta\left(\ast\mathbf{H}\right)+\mu^2\phi &= \mathbf{0}, \end{aligned} \tag{43}$$

where $\mathbf{H}\doteqdot\delta\phi$, the mass $\mu=const$ and the energy–momentum is given by

$$\mathbf{\Sigma}_{\alpha}=\mathbf{\Sigma}_{\alpha}^{[\phi]}+\mathbf{\Sigma}_{\alpha}^{[\mathbf{m}]},$$

$$\mathbf{\Sigma}_{\alpha}^{[\phi]}\doteqdot\frac{z_4k_0^2}{2\rho}\{(\mathbf{e}_{\alpha}\rfloor\ \mathbf{H})\wedge\ast\mathbf{H}-(\mathbf{e}_{\alpha}\rfloor\ast\mathbf{H})\wedge\mathbf{H}+\mu^2[(\mathbf{e}_{\alpha}\rfloor\ \phi)\wedge\ast\phi-(\mathbf{e}_{\alpha}\rfloor\ast\phi)\wedge\phi]\}$$

is the energy–momentum current of the Proca d–field and $\mathbf{\Sigma}_{\alpha}^{[\mu]}$ is the energy–momentum current of the additional matter d–fields satisfying the corresponding Euler–Largange equations.

The proof of the Theorem is just the reformulation with respect to N–adapted (co) frames (6) and (7) of similar considerations in Refs. [5, 20]. The constants $k_0,k_1....$ are taken in terms of the gravitational coupling constants like in [21] as to have connection to the usual MAG and Einstein theory for trivial N–connection structures and for the dimension $m\rightarrow 0$. We use the triplet ansatz sector (42) of MAG theories [5, 20]. It is a remarkable fact that the equivalence Theorem 3.2 holds also in presence of arbitrary N–connections i. e. for all type of anholonomic generalizations of the Einstein, Einstein–Cartan and Finsler–Lagrange and Cartan–Hamilton geometries by introducing canonical d–connections (we can also consider Berwald type d–connections).

Corollary 3.2. *In abstract index form, the effective field equations for the generalized Finsler–affine gravity following from (43) are written*

$$\begin{aligned}
\widehat{\mathbf{R}}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\overleftarrow{\widehat{\mathbf{R}}} &= \tilde{\kappa}\left(\mathbf{\Sigma}^{[\phi]}_{\alpha\beta} + \mathbf{\Sigma}^{[\mathbf{m}]}_{\alpha\beta}\right), \qquad (44)\\
\widehat{\mathbf{D}}_{\nu}\mathbf{H}^{\nu\mu} &= \mu^{2}\phi^{\mu},
\end{aligned}$$

with $\mathbf{H}_{\nu\mu} \doteqdot \widehat{\mathbf{D}}_{\nu}\phi_{\mu} - \widehat{\mathbf{D}}_{\mu}\phi_{\nu} + w^{\gamma}_{\mu\nu}\phi_{\gamma}$ *being the field strengths of the Abelian Proca field* $\phi^{\mu}, \tilde{\kappa} = const,$ *and*

$$\mathbf{\Sigma}^{[\phi]}_{\alpha\beta} = \mathbf{H}_{\alpha}^{\ \mu}\mathbf{H}_{\beta\mu} - \frac{1}{4}\mathbf{g}_{\alpha\beta}\mathbf{H}_{\mu\nu}\mathbf{H}^{\mu\nu} + \mu^{2}\phi_{\alpha}\phi_{\beta} - \frac{\mu^{2}}{2}\mathbf{g}_{\alpha\beta}\phi_{\mu}\phi^{\mu}. \qquad (45)$$

The Ricci d–tensor $\widehat{\mathbf{R}}_{\alpha\beta}$ and scalar $\overleftarrow{\widehat{\mathbf{R}}}$ from (44) can be decomposed in irreversible h– and v–invariant components like (31) and (32),

$$\begin{aligned}
\widehat{R}_{ij} - \frac{1}{2}g_{ij}\left(\widehat{R} + \widehat{S}\right) &= \tilde{\kappa}\left(\mathbf{\Sigma}^{[\phi]}_{ij} + \mathbf{\Sigma}^{[\mathbf{m}]}_{ij}\right), \qquad (46)\\
\widehat{S}_{ab} - \frac{1}{2}h_{ab}\left(\widehat{R} + \widehat{S}\right) &= \tilde{\kappa}\left(\mathbf{\Sigma}^{[\phi]}_{ab} + \mathbf{\Sigma}^{[\mathbf{m}]}_{ab}\right), \qquad (47)\\
{}^{1}P_{ai} &= \tilde{\kappa}\left(\mathbf{\Sigma}^{[\phi]}_{ai} + \mathbf{\Sigma}^{[\mathbf{m}]}_{ai}\right), \qquad (48)\\
-{}^{2}P_{ia} &= \tilde{\kappa}\left(\mathbf{\Sigma}^{[\phi]}_{ia} + \mathbf{\Sigma}^{[\mathbf{m}]}_{ia}\right). \qquad (49)
\end{aligned}$$

The constants are those from [5] being related to the constants from (37),

$$\mu^{2} = \frac{1}{z_{k}\kappa}\left(-4\beta_{4} + \frac{k_{1}}{2k_{0}}\beta_{5} + \frac{k_{2}}{k_{0}}\gamma_{4}\right),$$

where

$$\begin{aligned}
k_{0} &= 4\alpha_{2}\beta_{3} - 3(\gamma_{3})^{2} \neq 0,\ k_{1} = 9\left(\frac{1}{2}\alpha_{5}\beta_{5} - \gamma_{3}\gamma_{4}\right),\ k_{2} = 3\left(4\beta_{3}\gamma_{4} - \frac{3}{2}\beta_{5}\gamma_{3}\right),\\
\alpha_{2} &= a_{2} - 2a_{0},\ \beta_{3} = b_{3} + \frac{a_{0}}{8},\ \beta_{4} = b_{4} - \frac{3a_{0}}{8},\ \gamma_{3} = c_{3} + a_{0},\ \gamma_{4} = c_{4} + a_{0}.
\end{aligned}$$

If

$$\beta_{4} \rightarrow \frac{1}{4k_{0}}\left(\frac{1}{2}\beta_{5}k_{1} + k_{2}\gamma_{4}\right), \qquad (50)$$

the mass of Proca field $\mu^{2} \rightarrow 0$. The system becomes like the Einstein–Maxwell one with the source (45) defined by the antisymmetric field $\mathbf{H}_{\mu\nu}$ in its turn being determined by a solution of $\widehat{\mathbf{D}}_{\nu}\widehat{\mathbf{D}}^{\nu}\phi_{\alpha} = 0$ (a wave like equation in a curved space provided with N–connection). Even in this case the nonmetricity and torsion can be nontrivial, for instance, oscillating (see (42)).

We note that according the Remark 3.1, the system (44) defines, for instance, a Finsler configuration if the d–metric $\mathbf{g}_{\alpha\beta}$, the d–connection $\widehat{\mathbf{D}}_{\nu}$ and the N–connection are of Finsler type (or contains as imbeddings such objects).

3.3 Einstein–Cartan gravity and N–connections

The Einstein–Cartan gravity contains gravitational configurations with nontrivial N–connection structure. The simplest model with local anisotropy is to write on a space $\mathbf{V}^{n+m}$ the Einstein equations for the canonical d–connection $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$ (27) introduced in the Einstein d–tensor (33),

$$\widehat{\mathbf{R}}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\overleftarrow{\widehat{\mathbf{R}}} = \kappa\mathbf{\Sigma}^{[\mathbf{m}]}_{\alpha\beta},$$

or in terms of differential forms,

$$\eta_{\alpha\beta\gamma} \wedge \widehat{\mathbf{R}}^{\beta\gamma} = \kappa\mathbf{\Sigma}^{[\mathbf{m}]}_{\alpha} \tag{51}$$

which is a particular case of equations (43). The model contains nontrivial d–torsions, $\widehat{\mathbf{T}}^{\gamma}{}_{\alpha\beta}$, computed by introducing the components of (27) into formulas (29). We can consider that specific distributions of "spin dust/fluid" of Weyssenhoff and Raabe type, or any generalizations, adapted to the N–connection structure, can constitute the source of certain algebraic equations for torsion (see details in Refs. [19]) or even to consider generalizations for dynamical equations for torsion like in gauge gravity theories [22]. A more special case is defined by the theories when the d–torsions $\widehat{\mathbf{T}}^{\gamma}{}_{\alpha\beta}$ are induced by specific frame effects of N–connection structures. Such models contain all possible distorsions to generalized Finsler–Lagrange–Cartan spacetimes of the Einstein gravity and emphasize the conditions when such generalizations to locally anisotropic gravity preserve the local Lorentz invariance or even model Finsler like configurations in the framework of general relativity.

Let us express the 1–form of the canonical d–connection $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha}$ as the deformation of the Levi–Civita connection $\mathbf{\Gamma}^{\gamma}_{\bigtriangledown\ \alpha}$,

$$\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha} = \mathbf{\Gamma}^{\gamma}_{\bigtriangledown\ \alpha} + \widehat{\mathbf{Z}}^{\gamma}{}_{\alpha} \tag{52}$$

where

$$\widehat{\mathbf{Z}}_{\alpha\beta} = \mathbf{e}_{\beta}\rfloor\widehat{\mathbf{T}}_{\alpha} - \mathbf{e}_{\alpha}\rfloor\widehat{\mathbf{T}}_{\beta} + \frac{1}{2}\left(\mathbf{e}_{\alpha}\rfloor\mathbf{e}_{\beta}\rfloor\widehat{\mathbf{T}}_{\gamma}\right)\vartheta^{\gamma} \tag{53}$$

being a particular case of formulas (22) and (24) when nonmetricity vanishes, $\mathbf{Q}_{\alpha\beta} = 0$. This induces a distorsion of the curvature tensor like (41) but for d–objects, expressing (51) in the form

$$\eta_{\alpha\beta\gamma} \wedge \mathbf{R}^{\beta\gamma}_{\bigtriangledown} + \eta_{\alpha\beta\gamma} \wedge \mathbf{Z}^{\beta\gamma}_{\bigtriangledown} = \kappa\mathbf{\Sigma}^{[\mathbf{m}]}_{\alpha} \tag{54}$$

where

$$\mathbf{Z}^{\beta}_{\bigtriangledown\ \gamma} = \bigtriangledown\mathbf{Z}^{\beta}{}_{\gamma} + \mathbf{Z}^{\beta}{}_{\alpha} \wedge \mathbf{Z}^{\alpha}{}_{\gamma}.$$

Theorem 3.3. *The Einstein equations (51) for the canonical d–connection $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha}$ constructed for a d–metric field $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ (11) and N–connection N^a_i is equivalent to the gravitational field equations for the Einstein–Cartan theory with torsion $\widehat{\mathbf{T}}^{\gamma}{}_{\alpha}$ defined by the N–connection, see formulas (29).*

Proof: The proof is trivial and follows from decomposition (52).

Remark 3.2. *Every type of generalized Finsler–Lagrange geometries is characterized by a corresponding N– and d–connection and d–metric structures, see Tables 1–11 in Ref. [6]. For the canonical d–connection such locally anisotropic geometries can be modeled on Riemann–Cartan manifolds as solutions of (51) for a prescribed type of d–torsions (29).*

Corollary 3.3. *A generalized Finsler geometry can be modeled in a (pseudo) Riemann spacetime by a d–metric* $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ *(11), equivalently by generic off–diagonal metric (12), satisfying the Einstein equations for the Levi–Civita connection,*

$$\eta_{\alpha\beta\gamma} \wedge \mathbf{R}_{\bigtriangledown}^{\beta\gamma} = \kappa \mathbf{\Sigma}_{\alpha}^{[\mathbf{m}]} \tag{55}$$

if and only if

$$\eta_{\alpha\beta\gamma} \wedge \mathbf{Z}_{\bigtriangledown}^{\beta\gamma} = 0. \tag{56}$$

The proof follows from equations (54). We emphasize that the conditions (56) are imposed for the deformations of the Ricci tensors computed from distorsions of the Levi–Civita connection to the cannonical d–connection. In general, a solution $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ of the Einstein equations (55) can be characterized alternatively by d–connections and N–connections as follows from relation (28). The alternative geometric description contains nontrivial torsion fields. The simplest such anholonomic configurations can be defined by the condition of vanishing of N–connection curvature (5), $\Omega_{ij}^{a} = 0$, but even in such cases there are nontrivial anholonomy coefficients, see (9), $\mathbf{w}^{b}{}_{ia} = -\mathbf{w}^{b}{}_{ai} = \partial_a N_i^b$, and nonvanishing d–torsions (29),

$$\widehat{T}_{ja}^{i} = -\widehat{T}_{aj}^{i} = \widehat{C}_{.ja}^{i} \text{ and } \widehat{T}_{.bi}^{a} = -\widehat{T}_{.ib}^{a} = \widehat{P}_{.bi}^{a} = \frac{\partial N_i^a}{\partial y^b} - \widehat{L}_{.bj}^{a},$$

being induced by off–diagonal terms in the metric (12).

3.4 String gravity and N–connections

The subjects concerning generalized Finsler (super) geometry, spinors and (super) strings are analyzed in details in Refs. [9]. Here, we consider the simplest examples when Finsler like geometries can be modeled in string gravity and related to certain metric–affine structures.

For instance, in the sigma model for bosonic string (see, [1]), the background connection is taken to be not the Levi–Civita one, but a certain deformation by the strength (torsion) tensor

$$H_{\mu\nu\rho} \doteqdot \delta_\mu B_{\nu\rho} + \delta_\rho B_{\mu\nu} + \delta_\nu B_{\rho\mu}$$

of an antisymmetric field $B_{\nu\rho}$, defined as

$$\mathcal{D}_\mu = \bigtriangledown_\mu + \frac{1}{2} H_{\mu\nu}{}^{\rho}.$$

We consider the H–field defined by using N–elongated operators (6) in order to compute the coefficients with respect to anholonomic frames.

The condition of the Weyl invariance to hold in two dimensions in the lowest nontrivial approximation in string constant α', see [9], turn out to be

$$\begin{aligned}
R_{\mu\nu} &= -\frac{1}{4} H_{\mu}{}^{\nu\rho} H_{\nu\lambda\rho} + 2 \bigtriangledown_\mu \bigtriangledown_\nu \Phi, \\
\bigtriangledown_\lambda H^{\lambda}{}_{\mu\nu} &= 2 \left(\bigtriangledown_\lambda \Phi\right) H^{\lambda}{}_{\mu\nu}, \\
\left(\bigtriangledown \Phi\right)^2 &= \bigtriangledown_\lambda \bigtriangledown^{\lambda} \Phi + \frac{1}{4} R + \frac{1}{48} H_{\mu\nu\rho} H^{\mu\nu\rho}.
\end{aligned}$$

where Φ is the dilaton field. For trivial dilaton configurations, $\Phi = 0$, we may write

$$\begin{aligned} R_{\mu\nu} &= -\frac{1}{4} H_{\mu}^{\ \nu\rho} H_{\nu\lambda\rho}, \\ \bigtriangledown_{\lambda} H^{\lambda}_{\ \mu\nu} &= 0. \end{aligned}$$

In Refs. [9] we analyzed string gravity models derived from superstring effective actions, for instance, from the 4D Neveu-Schwarz action. In this paper we consider, for simplicity, a model with zero dilaton field but with nontrivial H–field related to the d–torsions induced by the N–connection and canonical d–connection.

A class of Finsler like metrics can be derived from the bosonic string theory if $\mathbf{H}_{\nu\lambda\rho}$ and $\mathbf{B}_{\nu\rho}$ are related to the d–torsions components, for instance, with $\widehat{\mathbf{T}}^{\gamma}_{\ \alpha\beta}$. Really, we can take an ansatz

$$\mathbf{B}_{\nu\rho} = [B_{ij}, B_{ia}, B_{ab}]$$

and consider that

$$\mathbf{H}_{\nu\lambda\rho} = \widehat{\mathbf{Z}}_{\ \nu\lambda\rho} + \widehat{\mathbf{H}}_{\nu\lambda\rho} \tag{57}$$

where $\widehat{\mathbf{Z}}_{\ \nu\lambda\rho}$ is the distorsion of the Levi–Civita connection induced by $\widehat{\mathbf{T}}^{\gamma}_{\ \alpha\beta}$, see (53). In this case the induced by N–connection torsion structure is related to the antisymmetric H–field and correspondingly to the B–field from string theory. The equations

$$\bigtriangledown^{\nu} \mathbf{H}_{\nu\lambda\rho} = \bigtriangledown^{\nu} (\widehat{\mathbf{Z}}_{\ \nu\lambda\rho} + \widehat{\mathbf{H}}_{\nu\lambda\rho}) = 0 \tag{58}$$

impose certain dynamical restrictions to the N–connection coefficients N_i^a and d–metric $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ contained in $\widehat{\mathbf{T}}^{\gamma}_{\ \alpha\beta}$. If on the background space it is prescribed the cannonical d–connection $\widehat{\mathbf{D}}$, we can state a model with (58) redefined as

$$\widehat{\mathbf{D}}^{\nu} \mathbf{H}_{\nu\lambda\rho} = \widehat{\mathbf{D}}^{\nu} (\widehat{\mathbf{Z}}_{\ \nu\lambda\rho} + \widehat{\mathbf{H}}_{\nu\lambda\rho}) = 0, \tag{59}$$

where $\widehat{\mathbf{H}}_{\nu\lambda\rho}$ are computed for stated values of $\widehat{\mathbf{T}}^{\gamma}_{\ \alpha\beta}$. For trivial N–connections when $\widehat{\mathbf{Z}}_{\ \nu\lambda\rho} \to 0$ and $\widehat{\mathbf{D}}^{\nu} \to \bigtriangledown^{\nu}$, the $\widehat{\mathbf{H}}_{\nu\lambda\rho}$ transforms into usual H–fields.

Proposition 3.1. *The dynamics of generalized Finsler–affine string gravity is defined by the system of field equations*

$$\begin{aligned} \widehat{\mathbf{R}}_{\alpha\beta} - \frac{1}{2} \mathbf{g}_{\alpha\beta} \overleftarrow{\widehat{\mathbf{R}}} &= \tilde{\kappa} \left(\mathbf{\Sigma}^{[\phi]}_{\alpha\beta} + \mathbf{\Sigma}^{[\mathbf{m}]}_{\alpha\beta} + \mathbf{\Sigma}^{[\mathbf{T}]}_{\alpha\beta} \right), \\ \widehat{\mathbf{D}}_{\nu} \mathbf{H}^{\nu\mu} &= \mu^2 \phi^{\mu}, \\ \widehat{\mathbf{D}}^{\nu} (\widehat{\mathbf{Z}}_{\ \nu\lambda\rho} + \widehat{\mathbf{H}}_{\nu\lambda\rho}) &= 0 \end{aligned} \tag{60}$$

with $\mathbf{H}_{\nu\mu} \doteqdot \widehat{\mathbf{D}}_{\nu} \phi_{\mu} - \widehat{\mathbf{D}}_{\mu} \phi_{\nu} + w^{\gamma}_{\mu\nu} \phi_{\gamma}$ *being the field strengths of the Abelian Proca field* ϕ^{μ}, $\tilde{\kappa} = const$,

$$\mathbf{\Sigma}^{[\phi]}_{\alpha\beta} = \mathbf{H}_{\alpha}^{\ \mu} \mathbf{H}_{\beta\mu} - \frac{1}{4} \mathbf{g}_{\alpha\beta} \mathbf{H}_{\mu\nu} \mathbf{H}^{\mu\nu} + \mu^2 \phi_{\alpha} \phi_{\beta} - \frac{\mu^2}{2} \mathbf{g}_{\alpha\beta} \phi_{\mu} \phi^{\mu},$$

and

$$\mathbf{\Sigma}^{[\mathbf{T}]}_{\alpha\beta} = \mathbf{\Sigma}^{[\mathbf{T}]}_{\alpha\beta} \left(\widehat{\mathbf{T}}, \Phi \right)$$

contains contributions of $\widehat{\mathbf{T}}$ *and* Φ *fields.*

Proof: It follows as an extension of the Corollary 3.2 to sources induced by string corrections. The system (60) should be completed by the field equations for the matter fields present in $\mathbf{\Sigma}^{[\mathbf{m}]}_{\alpha\beta}$.

Finally, we note that the equations (60) reduce to equations of type (54) (for Riemann–Cartan configurations with zero nonmetricity),

$$\eta_{\alpha\beta\gamma}\wedge\mathbf{R}^{\beta\gamma}_{\bigtriangledown}+\eta_{\alpha\beta\gamma}\wedge\mathbf{Z}^{\beta\gamma}_{\bigtriangledown}=\kappa\mathbf{\Sigma}^{[\mathbf{T}]}_{\alpha},$$

and to equations of type (55) and (56) (for (pseudo) Riemannian configurations)

$$\begin{aligned}\eta_{\alpha\beta\gamma}\wedge\mathbf{R}^{\beta\gamma}_{\bigtriangledown} &= \kappa\mathbf{\Sigma}^{[\mathbf{T}]}_{\alpha}, \\ \eta_{\alpha\beta\gamma}\wedge\mathbf{Z}^{\beta\gamma}_{\bigtriangledown} &= 0\end{aligned} \tag{61}$$

with sources defined by torsion (related to N–connection) from string theory.

4 The Anholonomic Frame Method in MAG and String Gravity

In a series of papers, see Refs. [7, 8, 10, 15], the anholonomic frame method of constructing exact solutions with generic off–diagonal metrics (depending on 2-4 variables) in general relativity, gauge gravity and certain extra dimension generalizations was elaborated. In this section, we develop the method in MAG and string gravity with applications to different models of five dimensional (in brief, 5D) generalized Finsler–affine spaces.

We consider a metric–affine space provided with N–connection structure $\mathbf{N}=[N^4_i(u^\alpha), N^5_i(u^\alpha)]$ where the local coordinates are labeled $u^\alpha=(x^i,y^4=v,y^5)$, for $i=1,2,3$. We state the general condition when exact solutions of the field equations of the generalized Finsler–affine string gravity depending on holonomic variables x^i and on one anholonomic (equivalently, anisotropic) variable $y^4=v$ can be constructed in explicit form. Every coordinate from a set u^α can may be time like, 3D space like, or extra dimensional. For simplicity, the partial derivatives are denoted $a^\times=\partial a/\partial x^1, a^\bullet=\partial a/\partial x^2, a'=\partial a/\partial x^3, a^*=\partial a/\partial v.$

The 5D metric

$$\mathbf{g}=\mathbf{g}_{\alpha\beta}\left(x^i,v\right)du^\alpha\otimes du^\beta \tag{62}$$

has the metric coefficients $\mathbf{g}_{\alpha\beta}$ parametrized with respect to the coordinate dual basis by an off–diagonal matrix (ansatz)

$$\left[\begin{array}{ccccc} g_1+{w_1}^2h_4+{n_1}^2h_5 & w_1w_2h_4+n_1n_2h_5 & w_1w_3h_4+n_1n_3h_5 & w_1h_4 & n_1h_5 \\ w_1w_2h_4+n_1n_2h_5 & g_2+{w_2}^2h_4+{n_2}^2h_5 & w_2w_3h_4+n_2n_3h_5 & w_2h_4 & n_2h_5 \\ w_1w_3h_4+n_1n_3h_5 & w_2w_3h_4+n_2n_3h_5 & g_3+{w_3}^2h_4+{n_3}^2h_5 & w_3h_4 & n_3h_5 \\ w_1h_4 & w_2h_4 & w_3h_4 & h_4 & 0 \\ n_1h_5 & n_2h_5 & n_3h_5 & 0 & h_5 \end{array}\right], \tag{63}$$

with the coefficients being some necessary smoothly class functions of type

$$\begin{aligned} g_1 &= \pm1, g_{2,3}=g_{2,3}(x^2,x^3), h_{4,5}=h_{4,5}(x^i,v), \\ w_i &= w_i(x^i,v), n_i=n_i(x^i,v), \end{aligned}$$

where the N–coefficients from (6) and (7) are parametrized $N^4_i=w_i$ and $N^5_i=n_i$.

Theorem 4.1. *The nontrivial components of the 5D Ricci d–tensors (31),* $\widehat{\mathbf{R}}_{\alpha\beta} = (\widehat{R}_{ij}, \widehat{R}_{ia}, \widehat{R}_{ai}, \widehat{S}_{ab})$*, for the d–metric (11) and canonical d–connection* $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$*(27) both defined by the ansatz (63), computed with respect to anholonomic frames (6) and (7), consist from h- and v–irreducible components:*

$$R_2^2 = R_3^3 = -\frac{1}{2g_2g_3}[g_3^{\bullet\bullet} - \frac{g_2^{\bullet}g_3^{\bullet}}{2g_2} - \frac{(g_3^{\bullet})^2}{2g_3} + g_2^{''} - \frac{g_2^{'}g_3^{'}}{2g_3} - \frac{(g_2^{'})^2}{2g_2}], \tag{64}$$

$$S_4^4 = S_5^5 = -\frac{1}{2h_4h_5}\left[h_5^{**} - h_5^{*}\left(\ln\sqrt{|h_4h_5|}\right)\right]^{*}\Big], \tag{65}$$

$$R_{4i} = -w_i\frac{\beta}{2h_5} - \frac{\alpha_i}{2h_5}, \tag{66}$$

$$R_{5i} = -\frac{h_5}{2h_4}\left[n_i^{**} + \gamma n_i^{*}\right], \tag{67}$$

where

$$\begin{aligned} \alpha_i &= \partial_i h_5^{*} - h_5^{*}\partial_i \ln\sqrt{|h_4h_5|}, \beta = h_5^{**} - h_5^{*}[\ln\sqrt{|h_4h_5|}]^{*}, \gamma = 3h_5^{*}/2h_5 - h_4^{*}/h_4 \\ h_4^{*} &\neq 0,\ h_5^{*} \neq 0 \ \ \textit{cases with vanishing}\ \ h_4^{*}\ \textit{and/or}\ \ h_5^{*}\ \textit{should be analyzed additionally.} \end{aligned} \tag{68}$$

The proof of Theorem 4.1 is given in Appendix A.

We can generalize the ansatz (63) by introducing a conformal factor $\omega(x^i, v)$ and additional deformations of the metric via coefficients $\zeta_{\hat{i}}(x^i, v)$ (here, the indices with 'hat' take values like $\hat{i} = 1, 2, 3, 5$), i. e. for metrics of type

$$\mathbf{g}^{[\omega]} = \omega^2(x^i, v)\hat{\mathbf{g}}_{\alpha\beta}\left(x^i, v\right) du^{\alpha} \otimes du^{\beta}, \tag{69}$$

were the coefficients $\hat{\mathbf{g}}_{\alpha\beta}$ are parametrized by the ansatz

$$\left[\begin{array}{ccccc} g_1 + ({w_1}^2 + {\zeta_1}^2)h_4 + {n_1}^2 h_5 & (w_1w_2 + \zeta_1\zeta_2)h_4 + n_1n_2h_5 & (w_1w_3 + \zeta_1\zeta_3)h_4 + n_1n_3h_5 & (w_1 + \zeta_1)h_4 & n_1h_5 \\ (w_1w_2 + \zeta_1\zeta_2)h_4 + n_1n_2h_5 & g_2 + ({w_2}^2 + {\zeta_2}^2)h_4 + {n_2}^2 h_5 & (w_2w_3 + +\zeta_2\zeta_3)h_4 + n_2n_3h_5 & (w_2 + \zeta_2)h_4 & n_2h_5 \\ (w_1w_3 + \zeta_1\zeta_3)h_4 + n_1n_3h_5 & (w_2w_3 + +\zeta_2\zeta_3)h_4 + n_2n_3h_5 & g_3 + ({w_3}^2 + {\zeta_3}^2)h_4 + {n_3}^2 h_5 & (w_3 + \zeta_3)h_4 & n_3h_5 \\ (w_1 + \zeta_1)h_4 & (w_2 + \zeta_2)h_4 & (w_3 + \zeta_3)h_4 & h_4 & 0 \\ n_1h_5 & n_2h_5 & n_3h_5 & 0 & h_5 + \zeta_5h_4 \end{array}\right]. \tag{70}$$

Such 5D metrics have a second order anisotropy [9, 13] when the N–coefficients are parametrized in the first order anisotropy like $N_i^4 = w_i$ and $N_i^5 = n_i$ (with three anholonomic, x^i, and two anholonomic, y^4 and y^5, coordinates) and in the second order anisotropy (on the second 'shell', with four holonomic, (x^i, y^5), and one anholonomic,y^4, coordinates) with $N_{\hat{i}}^5 = \zeta_{\hat{i}}$, in this work we state, for simplicity, $\zeta_5 = 0$. For trivial values $\omega = 1$ and $\zeta_{\hat{i}} = 0$, the metric (69) transforms into (62).

The Theorem 4.1 can be extended as to include the ansatz (69):

Theorem 4.2. *The nontrivial components of the 5D Ricci d–tensors (31),* $\widehat{\mathbf{R}}_{\alpha\beta} = (\widehat{R}_{ij}, \widehat{R}_{ia}, \widehat{R}_{ai}, \widehat{S}_{ab})$*, for the metric (11) and canonical d–connection* $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$ *(27) defined by the ansatz (4), computed with respect to the anholonomic frames (6) and (7), are given by the same formulas (64)–(67) if there are satisfied the conditions*

$$\hat{\delta}_i h_4 = 0 \ \textit{and}\ \hat{\delta}_i \omega = 0 \tag{71}$$

for $\hat{\delta}_i = \partial_i - (w_i + \zeta_i)\,\partial_4 + n_i\partial_5$ *when the values* $\zeta_{\tilde{i}} = (\zeta_i, \zeta_5 = 0)$ *are to be defined as any solutions of (71).*

The proof of Theorem 4.2 consists from a straightforward calculation of the components of the Ricci tensor (31) like in Appendix A. The simplest way to do this is to compute the deformations by the conformal factor of the coefficients of the canonical connection (27) and then to use the calculus for Theorem 4.1. Such deformations induce corresponding deformations of the Ricci tensor (31). The condition that we have the same values of the Ricci tensor for the (12) and (4) results in equations (71) and (73) which are compatible, for instance, if for instance, if

$$\omega^{q_1/q_2} = h_4 \ (q_1 \text{ and } q_2 \text{ are integers}), \tag{72}$$

and ζ_i satisfy the equations

$$\partial_i \omega - (w_i + \zeta_i)\omega^* = 0. \tag{73}$$

There are also different possibilities to satisfy the condition (71). For instance, if $\omega = \omega_1$ ω_2, we can consider that $h_4 = \omega_1^{q_1/q_2}\ \omega_2^{q_3/q_4}$ for some integers q_1, q_2, q_3 and q_4■

There are some important consequences of the Theorems 4.1 and 4.2:

Corollary 4.1. *The non–trivial components of the Einstein tensor [see (33) for the canonical d–connection]* $\widehat{\mathbf{G}}^{\alpha}{}_{\beta} = \widehat{\mathbf{R}}^{\alpha}{}_{\beta} - \frac{1}{2}\overleftarrow{\widehat{\mathbf{R}}}\delta^{\alpha}_{\beta}$ *for the ansatz (63) and (4) given with respect to the N–adapted (co) frames are*

$$G^1_1 = -\left(R^2_2 + S^4_4\right), G^2_2 = G^3_3 = -S^4_4, G^4_4 = G^5_5 = -R^2_2. \tag{74}$$

The relations (74) can be derived following the formulas for the Ricci tensor (64)–(67). They impose the condition that the dynamics of such gravitational fields is defined by two independent components R^2_2 and S^4_4 and result in

Corollary 4.2. *The system of effective 5D Einstein–Proca equations on spaces provided with N–connection structure (44) (equivalently, the system (46)–(49) is compatible for the generic off–diagonal ansatz (63) and (4) if the energy–momentum tensor* $\mathbf{\Upsilon}_{\alpha\beta} = \tilde{\kappa}(\mathbf{\Sigma}^{[\phi]}_{\alpha\beta} + \mathbf{\Sigma}^{[\mathbf{m}]}_{\alpha\beta})$ *of the Proca and matter fields given with respect to N– frames is diagonal and satisfies the conditions*

$$\Upsilon^2_2 = \Upsilon^3_3 = \Upsilon_2(x^2, x^3, v),\ \Upsilon^4_4 = \Upsilon^5_5 = \Upsilon_4(x^2, x^3),\ \text{and } \Upsilon_1 = \Upsilon_2 + \Upsilon_4. \tag{75}$$

Remark 4.1. *Instead of the energy–momentum tensor* $\mathbf{\Upsilon}_{\alpha\beta} = \tilde{\kappa}(\mathbf{\Sigma}^{[\phi]}_{\alpha\beta} + \mathbf{\Sigma}^{[\mathbf{m}]}_{\alpha\beta})$ *for the Proca and matter fields we can consider any source, for instance, with string corrections, when* $\mathbf{\Upsilon}^{[str]}_{\alpha\beta} = \tilde{\kappa}\left(\mathbf{\Sigma}^{[\phi]}_{\alpha\beta} + \mathbf{\Sigma}^{[\mathbf{m}]}_{\alpha\beta} + \mathbf{\Sigma}^{[\mathbf{T}]}_{\alpha\beta}\right)$ *like in (60) satisfying the conditions (75).*

If the conditions of the Corollary 4.2, or Remark 4.1, are satisfied, the h- and v– irreducible components of the 5D Einstein–Proca equations (46) and (49), or of the string gravity equations (60), for the ansatz (63) and (4) transform into the system

$$R^2_2 = R^3_3 = -\frac{1}{2g_2g_3}[g_3^{\bullet\bullet} - \frac{g_2^{\bullet}g_3^{\bullet}}{2g_2} - \frac{(g_3^{\bullet})^2}{2g_3} + g_2^{''} - \frac{g_2^{'}g_3^{'}}{2g_3} - \frac{(g_2^{'})^2}{2g_2}] = -\Upsilon_4(x^2, x^3), \tag{76}$$

$$S^4_4 = S^5_5 = -\frac{1}{2h_4h_5}\left[h_5^{**} - h_5^{*}\left(\ln\sqrt{|h_4h_5|}\right)^{*}\right] = -\Upsilon_2(x^2, x^3, v). \tag{77}$$

$$R_{4i} = -w_i\frac{\beta}{2h_5} - \frac{\alpha_i}{2h_5} = 0, \tag{78}$$

$$R_{5i} = -\frac{h_5}{2h_4}\left[n_i^{**} + \gamma n_i^{*}\right] = 0. \tag{79}$$

A very surprising result is that we are able to construct exact solutions of the 5D Einstein–Proca equations with anholonomic variables and generic off–diagonal metrics:

Theorem 4.3. *The system of second order nonlinear partial differential equations (76)–(79) and (73) can be solved in general form if there are given certain values of functions $g_2(x^2,x^3)$ (or, inversely, $g_3(x^2,x^3)$), $h_4(x^i,v)$ (or, inversely, $h_5(x^i,v)$), $\omega(x^i,v)$ and of sources $\Upsilon_2(x^2,x^3,v)$ and $\Upsilon_4(x^2,x^3)$.*

We outline the main steps of constructing exact solutions and proving this Theorem.

- The general solution of equation (76) can be written in the form

$$\varpi = g_{[0]} \exp[a_2 \widetilde{x}^2 \left(x^2,x^3\right) + a_3 \widetilde{x}^3 \left(x^2,x^3\right)], \tag{80}$$

  were $g_{[0]}, a_2$ and a_3 are some constants and the functions $\widetilde{x}^{2,3}\left(x^2,x^3\right)$ define any coordinate transforms $x^{2,3} \rightarrow \widetilde{x}^{2,3}$ for which the 2D line element becomes conformally flat, i. e.

$$g_2(x^2,x^3)(dx^2)^2 + g_3(x^2,x^3)(dx^3)^2 \rightarrow \varpi(x^2,x^3)\left[(d\widetilde{x}^2)^2 + \epsilon(d\widetilde{x}^3)^2\right], \tag{81}$$

  where $\epsilon = \pm 1$ for a corresponding signature. In coordinates $\widetilde{x}^{2,3}$, the equation (76) transform into

$$\varpi\left(\ddot{\varpi} + \varpi''\right) - \dot{\varpi} - \varpi' = 2\varpi^2 \Upsilon_4(\tilde{x}^2,\tilde{x}^3)$$

  or

$$\ddot{\psi} + \psi'' = 2\Upsilon_4(\tilde{x}^2,\tilde{x}^3), \tag{82}$$

  for $\psi = \ln|\varpi|$. The integrals of (82) depends on the source Υ_4. As a particular case we can consider that $\Upsilon_4 = 0$. There are three alternative possibilities to generate solutions of (76). For instance, we can prescribe that $g_2 = g_3$ and get the equation (82) for $\psi = \ln|g_2| = \ln|g_3|$. If we suppose that $g_2^{'} = 0$, for a given $g_2(x^2)$, we obtain from (76)

$$g_3^{\bullet\bullet} - \frac{g_2^{\bullet} g_3^{\bullet}}{2g_2} - \frac{(g_3^{\bullet})^2}{2g_3} = 2g_2 g_3 \Upsilon_4(x^2,x^3)$$

  which can be integrated explicitly for given values of Υ_4. Similarly, we can generate solutions for a prescribed $g_3(x^3)$ in the equation

$$g_2^{''} - \frac{g_2^{'} g_3^{'}}{2g_3} - \frac{(g_2^{'})^2}{2g_2} = 2g_2 g_3 \Upsilon_4(x^2,x^3).$$

  We note that a transform (81) is always possible for 2D metrics and the explicit form of solutions depends on chosen system of 2D coordinates and on the signature $\epsilon = \pm 1$. In the simplest case with $\Upsilon_4 = 0$ the equation (76) is solved by arbitrary two functions $g_2(x^3)$ and $g_3(x^2)$.

- For $\Upsilon_2 = 0$, the equation (77) relates two functions $h_4(x^i,v)$ and $h_5(x^i,v)$ following two possibilities:

a) to compute

$$\begin{aligned}\sqrt{|h_5|} &= h_{5[1]}\left(x^i\right) + h_{5[2]}\left(x^i\right) \int \sqrt{|h_4\left(x^i,v\right)|}dv,\ h_4^*\left(x^i,v\right) \neq 0; \\ &= h_{5[1]}\left(x^i\right) + h_{5[2]}\left(x^i\right) v,\ h_4^*\left(x^i,v\right) = 0, \end{aligned} \tag{83}$$

for some functions $h_{5[1,2]}\left(x^i\right)$ stated by boundary conditions;

b) or, inversely, to compute h_4 for a given $h_5\left(x^i,v\right), h_5^* \neq 0$,

$$\sqrt{|h_4|} = h_{[0]}\left(x^i\right) (\sqrt{|h_5\left(x^i,v\right)|})^*, \tag{84}$$

with $h_{[0]}\left(x^i\right)$ given by boundary conditions. We note that the sourceless equation (77) is satisfied by arbitrary pairs of coefficients $h_4\left(x^i,v\right)$ and $h_{5[0]}\left(x^i\right)$. Solutions with $\Upsilon_2 \neq 0$ can be found by ansatz of type

$$h_5[\Upsilon_2] = h_5, h_4[\Upsilon_2] = \varsigma_4\left(x^i,v\right) h_4, \tag{85}$$

where h_4 and h_5 are related by formula (83), or (84). Substituting (85), we obtain

$$\varsigma_4\left(x^i,v\right) = \varsigma_{4[0]}\left(x^i\right) - \int \Upsilon_2(x^2,x^3,v)\frac{h_4 h_5}{4h_5^*}dv, \tag{86}$$

where $\varsigma_{4[0]}\left(x^i\right)$ are arbitrary functions.

- The exact solutions of (78) for $\beta \neq 0$ are defined from an algebraic equation, $w_i\beta + \alpha_i = 0$, where the coefficients β and α_i are computed as in formulas (68) by using the solutions for (76) and (77). The general solution is

$$w_k = \partial_k \ln[\sqrt{|h_4 h_5|}/|h_5^*|]/\partial_v \ln[\sqrt{|h_4 h_5|}/|h_5^*|], \tag{87}$$

with $\partial_v = \partial/\partial v$ and $h_5^* \neq 0$. If $h_5^* = 0$, or even $h_5^* \neq 0$ but $\beta = 0$, the coefficients w_k could be arbitrary functions on $\left(x^i,v\right)$. For the vacuum Einstein equations this is a degenerated case imposing the the compatibility conditions $\beta = \alpha_i = 0$, which are satisfied, for instance, if the h_4 and h_5 are related as in the formula (84) but with $h_{[0]}\left(x^i\right) = const$.

- Having defined h_4 and h_5 and computed γ from (68) we can solve the equation (79) by integrating on variable "v" the equation $n_i^{**} + \gamma n_i^* = 0$. The exact solution is

$$\begin{aligned} n_k &= n_{k[1]}\left(x^i\right) + n_{k[2]}\left(x^i\right) \int [h_4/(\sqrt{|h_5|})^3]dv,\ h_5^* \neq 0; \\ &= n_{k[1]}\left(x^i\right) + n_{k[2]}\left(x^i\right) \int h_4 dv, \qquad h_5^* = 0; \\ &= n_{k[1]}\left(x^i\right) + n_{k[2]}\left(x^i\right) \int [1/(\sqrt{|h_5|})^3]dv,\ h_4^* = 0, \end{aligned} \tag{88}$$

for some functions $n_{k[1,2]}\left(x^i\right)$ stated by boundary conditions.

The exact solution of (73) is given by some arbitrary functions $\zeta_i = \zeta_i(x^i, v)$ if both $\partial_i\omega = 0$ and $\omega^* = 0$, we chose $\zeta_i = 0$ for $\omega = const$, and

$$\begin{aligned} \zeta_i &= -w_i + (\omega^*)^{-1}\partial_i\omega, \quad \omega^* \neq 0, \\ &= (\omega^*)^{-1}\partial_i\omega, \quad \omega^* \neq 0, \text{ for vacuum solutions.} \end{aligned} \tag{89}$$

The Theorem 4.3 states a general method of constructing exact solutions in MAG, of the Einstein–Proca equations and various string gravity generalizations with generic off–diagonal metrics. Such solutions are with associated N–connection structure. This method can be also applied in order to generate, for instance, certain Finsler or Lagrange configurations as v-irreducible components. The 5D ansatz can not be used to generate standard Finsler or Lagrange geometries because the dimension of such spaces can not be an odd number. Nevertheless, the anholonomic frame method can be applied in order to generate 4D exact solutions containing Finsler–Lagrange configurations, see Appendix B.

Summarizing the results for the nondegenerated cases when $h_4^* \neq 0$ and $h_5^* \neq 0$ and (for simplicity, for a trivial conformal factor ω), we derive an explicit result for 5D exact solutions with local coordinates $u^\alpha = (x^i, y^a)$ when $x^i = \left(x^1, x^{\widehat{i}}\right), x^{\widehat{i}} = (x^2, x^3), y^a = (y^4 = v, y^a)$ and arbitrary signatures $\epsilon_\alpha = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5)$ (where $\epsilon_\alpha = \pm 1$) :

Corollary 4.3. *Any off–diagonal metric*

$$\begin{aligned} \delta s^2 &= \epsilon_1 (dx^1)^2 + \epsilon_{\widehat{k}} g_{\widehat{k}}\left(x^{\widehat{i}}\right)(dx^{\widehat{k}})^2 + \\ & \quad \epsilon_4 h_0^2(x^i)\left[f^*\left(x^i, v\right)\right]^2 |\varsigma_\Upsilon\left(x^i, v\right)| (\delta v)^2 + \epsilon_5 f^2\left(x^i, v\right)\left(\delta y^5\right)^2, \\ \delta v &= dv + w_k\left(x^i, v\right) dx^k, \ \delta y^5 = dy^5 + n_k\left(x^i, v\right) dx^k, \end{aligned} \tag{90}$$

with coefficients of necessary smooth class, where $g_{\widehat{k}}\left(x^{\widehat{i}}\right)$ *is a solution of the 2D equation (76) for a given source* $\Upsilon_4\left(x^{\widehat{i}}\right)$,

$$\varsigma_\Upsilon\left(x^i, v\right) = \varsigma_4\left(x^i, v\right) = \varsigma_{4[0]}\left(x^i\right) - \frac{\epsilon_4}{16} h_0^2(x^i) \int \Upsilon_2(x^{\widehat{k}}, v)[f^2\left(x^i, v\right)]^2 dv,$$

and the N–connection coefficients $N_i^4 = w_i(x^k, v)$ *and* $N_i^5 = n_i(x^k, v)$ *are*

$$w_i = -\frac{\partial_i \varsigma_\Upsilon\left(x^k, v\right)}{\varsigma_\Upsilon^*\left(x^k, v\right)} \tag{91}$$

and

$$n_k = n_{k[1]}\left(x^i\right) + n_{k[2]}\left(x^i\right) \int \frac{\left[f^*\left(x^i, v\right)\right]^2}{\left[f\left(x^i, v\right)\right]^2} \varsigma_\Upsilon\left(x^i, v\right) dv, \tag{92}$$

define an exact solution of the system of Einstein equations with holonomic and anholonomic variables (76)–(79) for arbitrary nontrivial functions $f\left(x^i, v\right)$ *(with* $f^* \neq 0$*),* $h_0^2(x^i)$, $\varsigma_{4[0]}\left(x^i\right), n_{k[1]}\left(x^i\right)$ *and* $n_{k[2]}\left(x^i\right)$, *and sources* $\Upsilon_2(x^{\widehat{k}}, v), \Upsilon_4\left(x^{\widehat{i}}\right)$ *and any integration constants and signatures* $\epsilon_\alpha = \pm 1$ *to be defined by certain boundary conditions and physical considerations.*

Any metric (90) with $h_4^* \neq 0$ and $h_5^* \neq 0$ has the property to be generated by a function of four variables $f(x^i, v)$ with emphasized dependence on the anisotropic coordinate v, because $f^* \doteqdot \partial_v f \neq 0$ and by arbitrary sources $\Upsilon_2(x^{\widehat{k}}, v), \Upsilon_4\left(x^{\widehat{i}}\right)$. The rest of arbitrary functions not depending on v have been obtained in result of integradion of partial differential equations. This fix a specific class of metrics generated by using the relation (84) and the first formula in (88). We can generate also a different class of solutions with $h_4^* = 0$ by considering the second formula in (83) and respective formulas in (88). The "degenerated" cases with $h_4^* = 0$ but $h_5^* \neq 0$ and inversely, $h_4^* \neq 0$ but $h_5^* = 0$ are more special and request a proper explicit construction of solutions. Nevertheless, such type of solutions are also generic off–diagonal and they could be of substantial interest.

The sourceless case with vanishing Υ_2 and Υ_4 is defined following

Remark 4.2. *Any off–diagonal metric (90) with $\varsigma_\Upsilon = 1$, $h_0^2(x^i) = h_0^2 = const$, $w_i = 0$ and n_k computed as in (92) but for $\varsigma_\Upsilon = 1$, defines a vacuum solution of 5D Einstein equations for the canonical d–connection (27) computed for the ansatz (90).*

By imposing additional constraints on arbitrary functions from $N_i^5 = n_i$ and $N_i^5 = w_i$, we can select off–diagonal gravitational configurations with distorsions of the Levi–Civita connection resulting in canonical d–connections with the same solutions of the vacuum Einstein equations. For instance, we can model Finsler like geometries in general relativity, see Corollary 3.3. Under similar conditions the ansatz (63) was used for constructing exact off–diagonal solutions in the 5D Einstein gravity, see Refs. [7, 8, 9].

Let us consider the procedure of selecting solutions with off–diagonal metrics from an ansatz (90) with trivial N–connection curvature (such metrics consists a simplest subclass which can be restricted to (pseudo) Riemannian ones). The corresponding nontrivial coefficients the N–connection curvature (5) are computed

$$\Omega_{ij}^4 = \partial_i w_j - \partial_j w_i + w_i w_j^* - w_j w_i^* \text{ and } \Omega_{ij}^5 = \partial_i n_j - \partial_j n_i + w_i n_j^* - w_j n_i^*.$$

So, there are imposed six constraints, $\Omega_{ij}^4 = \Omega_{ij}^5 = 0$, for $i, j... = 1, 2, 4$ on six functions w_i and n_i computed respectively as (92) and (92) which can be satisfied by a corresponding subclass of functions $f(x^i, v)$ (with $f^* \neq 0$), $h_0^2(x^i)$, $\varsigma_{4[0]}(x^i), n_{k[1]}(x^i)$, $n_{k[2]}(x^i)$ and $\Upsilon_2(x^{\widehat{k}}, v), \Upsilon_4\left(x^{\widehat{i}}\right)$ (in general, we have to solve certain first order partial derivative equations with may be reduced to algebraic relations by corresponding parametrizations). For instance, in the vacuum case when $w_j = 0$, we obtain $\Omega_{ij}^5 = \partial_i n_j - \partial_j n_i$. The simplest example when condition $\Omega_{\widehat{i}\widehat{j}}^5 = \partial_{\widehat{i}} n_{\widehat{j}} - \partial_{\widehat{j}} n_{\widehat{i}} = 0$, with $\widehat{i}, \widehat{j} = 2, 3$ (reducing the metric (90) to a 4D one trivially embedded into 5D) is satisfied is to take $n_{3[1]} = n_{3[2]} = 0$ in (92) and consider that $f = f(x^2, v)$ with $n_{2[1]} = n_{2[1]}(x^2)$ and $n_{2[2]} = n_{2[2]}(x^2)$, i. e. by eliminating the dependence of the coefficients on x^3. This also results in a generic off–diagonal solution, because the anholonomy coefficients (9) are not trivial, for instance, $w_{24}^5 = n_2^*$ and $w_{14}^5 = n_1^*$.

Another interesting remark is that even we have reduced the canonical d–connection to the Levi–Civita one [with respect to N–adapted (co) frames; this imposes the metric to be (pseudo) Riemannian] by selecting the arbitrary functions as to have $\Omega_{ij}^a = 0$, one could be nonvanishing d–torsion components like $T_{41}^5 = P_{41}^5$ and $T_{41}^5 = P_{41}^5$ in (29). Such objects, as well the anholonomy coefficients w_{24}^5 and w_{14}^5 (which can be also considered

as torsion like objects) are constructed by taking certain "scarps" from the coefficients of off–diagonal metrics and anholonomic frames. They are induced by the frame anholonomy (like "torsions" in rotating anholonomic systems of reference for the Newton gravtity and mechanics with constraints) and vanish if we transfer the constructions with respect to any holonomic basis.

The above presented results are for generic 5D off–diagonal metrics, anholonomic transforms and nonlinear field equations. Reductions to a lower dimensional theory are not trivial in such cases. We emphasize some specific points of this procedure in the Appendix B (see details in [15]).

5 Exact Solutions

There were found a set of exact solutions in MAG [16, 20, 5] describing various configuration of Einstein–Maxwell of dilaton gravity emerging from low energy string theory, soliton and multipole solutions and generalized Plebanski–Demianski solutions, colliding waves and static black hole metrics. In this section we are going to look for some classes of 4D and 5D solutions of the Einstein–Proca equations in MAG related to string gravity modeling generalized Finsler–affine geometries and extending to such spacetimes some our previous results [7, 8, 9].

5.1 Finsler–Lagrange metrics in string and metric–affine gravity

As we discussed in section 2, the generalized Finsler–Lagrange spaces can be modeled in metric–affine spacetimes provided with N–connection structure. In this subsection, we show how such two dimensional Finsler like spaces with d–metrics depending on one anisotropic coordinate $y^4 = v$ (denoted as $\mathbf{F}^2 = \left[V^2, F\left(x^2, x^3, y\right)\right]$, $\mathbf{L}^2 = \left[V^2, L\left(x^2, x^3, y\right)\right]$ and $\mathbf{GL}^2 = \left[V^2, g_{ij}\left(x^2, x^3, y\right)\right]$ according to Ref. [6]) can be modeled by corresponding diad transforms on spacetimes with 5D (or 4D) d–metrics being exact solutions of the field equations for the generalized Finsler–affine string gravity (60) (as a particular case we can consider the Einstein–Proca system (76)–(79) and (73)). For every particular case of locally anisotropic spacetime, for instance, outlined in Appendix C, see Table 1, the quadratic form $\tilde{g}_{ij}$, d–metric $\tilde{\mathbf{g}}_{\alpha\beta} = [\tilde{g}_{ij}, \tilde{g}_{ij}]$ and N–connection $\tilde{N}^a_j$ one holds

Theorem 5.1. *Any 2D locally anisotropic structure given by* $\tilde{\mathbf{g}}_{\alpha\beta}$ *and* $\tilde{N}^a_j$ *can be modeled on the space of exact solutions of the 5D (or 4D) the generalized Finsler–affine string gravity system defined by the ansatz (4) (or (126)).*

We give the proof via an explicit construction. Let us consider

$$\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}] = \left[\omega g_2\left(x^2, x^3\right), \omega g_3\left(x^2, x^3\right), \omega h_4\left(x^2, x^3, v\right), \omega h_5\left(x^2, x^3, v\right)\right]$$

for $\omega = \omega\left(x^2, x^3, v\right)$ and

$$N^a_i = \left[N^4_i = w_i\left(x^2, x^3, v\right), N^5_i = n_i\left(x^2, x^3, v\right)\right],$$

where indices are running the values $a = 4, 5$ and $i = 2, 3$ define an exact 4D solution of the equations (60) (or, in the particular case, of the system (76)–(79), for simplicity, we put

$\omega(x^2, x^3, v) = 1$). We can relate the data $(\mathbf{g}_{\alpha\beta}, N_i^a)$ to any data $\left(\tilde{\mathbf{g}}_{\alpha\beta}, \tilde{N}_j^a\right)$ via nondegenerate diadic transforms $e_i^{i'} = e_i^{i'}(x^2, x^3, v)$, $l_a^{i'} = l_a^{i'}(x^2, x^3, v)$ and $q_a^{i'} = q_a^{i'}(x^2, x^3, v)$ (and theirs inverse matrices)

$$g_{ij} = e_i^{i'} e_j^{j'} \tilde{g}_{i'j'}, \ h_{ab} = l_a^{i'} l_b^{i'} \tilde{g}_{i'j'}, N_{i'}^a = q_{a'}^a \tilde{N}_{j'}^{a'} \ . \tag{93}$$

Such transforms may be associated to certain tetradic transforms of the N–elongated (co) frames ((7)) (6). If for the given data $(\mathbf{g}_{\alpha\beta}, N_i^a)$ and $\left(\tilde{\mathbf{g}}_{\alpha\beta}, \tilde{N}_j^a\right)$ in (93), we can solve the corresponding systems of quadratic algebraic equations and define nondegenerate matrices $\left(e_i^{i'}\right), \left(l_a^{i'}\right)$ and $\left(q_{i'}^a\right)$, we argue that the 2D locally anisotropic spacetime $\left(\tilde{\mathbf{g}}_{\alpha\beta}, \tilde{N}_j^a\right)$ (really, it is a 4D spacetime with generic off–diagonal metric and associated N–connection structure) can be modeled on by a class of exact solutions of effective Einstein–Proca equations for MAG.■

The d–metric with respect to transformed N–adapted diads is written in the form

$$\mathbf{g} = \tilde{g}_{i'j'} \mathbf{e}^{i'} \otimes \mathbf{e}^{j'} + \tilde{g}_{i'j'} \tilde{\mathbf{e}}^{i'} \otimes \tilde{\mathbf{e}}^{j'} \tag{94}$$

where

$$\mathbf{e}^{i'} = e_i^{i'} dx^i, \ \tilde{\mathbf{e}}^{i'} = l_a^{i'} \tilde{\mathbf{e}}^a, \ \tilde{\mathbf{e}}^a = dy^a + \tilde{N}_{j'}^a \tilde{\mathbf{e}}_{[N]}^{j'}, \ \tilde{\mathbf{e}}_{[N]}^{j'} = q_i^{j'} dx^i.$$

The d–metric (94) has the coefficients corresponding to generalized Finsler–Lagange spaces and emphasizes that any quadratic form $\tilde{g}_{i'j'}$ from Table 1 can be related via an exact solution $(g_{ij}, h_{ab}, N_{i'}^a)$.

We note that we can define particular cases of imbeddings with $h_{ab} = l_a^{i'} l_b^{i'} \tilde{g}_{i'j'}$ and $N_{j'}^a = q_{i'}^a \tilde{N}_{j'}^{i'}$ for a prescribed value of $g_{ij} = \tilde{g}_{i'j'}$ and try to model only the quadratic form $\tilde{h}_{i'j'}$ in MAG. Similar considerations were presented for particular cases of modeling Finsler structures and generalizations in Einstein and Einstein–Cartan spaces [7, 9], see the conditions (61).

5.2 Solutions in MAG with effective variable cosmological constant

A class of 4D solutions in MAG with local anisotropy can be derived from (44) for $\mathbf{\Sigma}_{\alpha\beta}^{[\mathbf{m}]} = 0$ and almost vanishing mass $\mu \to 0$ of the Proca field in the source $\mathbf{\Sigma}_{\alpha\beta}^{[\phi]}$. This holds in absence of matter fields and when the constant in the action for the Finsler–affine gravity are subjected to the condition (50). We consider that $\phi_\mu = \left(\phi_{\widehat{i}}\left(x^{\widehat{k}}\right), \phi_a = 0\right)$, where $\widehat{i}, \widehat{k}, ... = 2, 3$ and $a, b, ... = 4, 5$, with respect to a N–adapted coframe (7) and choose a metric ansatz of type (124) with $g_2 = 1$ and $g_3 = -1$ which select a flat h–subspace imbedded into a general anholonomic 4D background with nontrivial h_{ab} and N–connection structure N_i^a. The h–covariant derivatives are $\widehat{D}^{[h]} \phi_{\widehat{i}} = (\partial_2 \phi_{\widehat{i}}, \partial_3 \phi_{\widehat{i}})$ because the coefficients $\widehat{L}^i{}_{jk}$ and $\widehat{C}^i{}_{ja}$ are zero in (27) and any contraction with $\phi_a = 0$ results in zero values. In this case the Proca equations, $\widehat{\mathbf{D}}_\nu \mathbf{H}^{\nu\mu} = \mu^2 \phi^\mu$, transform in a Maxwell like equation,

$$\partial_2(\partial_2 \phi_{\widehat{i}}) - \partial_3(\partial_3 \phi_{\widehat{i}}) = 0, \tag{95}$$

for the potential $\phi_{\widehat{i}}$, with the dynamics in the h–subspace distinguished by a N–connection structure to be defined latter. We note that ϕ_i is not an electromagnetic field, but a component of the metric–affine gravity related to nonmetricity and torsion. The relation $\mathbf{Q} = k_0\phi,\ \mathbf{\Lambda} = k_1\phi,\ \mathbf{T} = k_2\phi$ from (42) transforms into $Q_{\widehat{i}} = k_0\phi_{\widehat{i}}, \Lambda_{\widehat{i}} = k_1\phi_{\widehat{i}}, T_i = k_2\phi_{\widehat{i}}$, and vanishing Q_a, Λ_a and T_a, defined, for instance, by a wave solution of (95),

$$\phi_{\widehat{i}} = \phi_{[0]\widehat{i}} \cos\left(\varrho_i x^i + \varphi_{[0]}\right) \tag{96}$$

for any constants $\phi_{[0]2,3}$, $\varphi_{[0]}$ and $(\varrho_2)^2 - (\varrho_3)^2 = 0$. In this simplified model we have related plane waves of nonmetricity and torsion propagating on an anholonomic background provided with N–connection. Such nonmetricity and torsion do not vanish even $\mu \to 0$ and the Proca field is approximated by a massless vector field defined in the h–subspace.

The energy–momentum tensor $\mathbf{\Sigma}^{[\phi]}_{\alpha\beta}$ for the massless field (96) is defined by a nontrivial value

$$H_{23} = \partial_2\phi_3 - \partial_3\phi_2 = \varepsilon_{23}\lambda_{[h]} \sin\left(\varrho_i x^i + \varphi_{[0]}\right)$$

with antisymmetric $\varepsilon_{23}, \varepsilon_{23} = 1$, and constant $\lambda_{[h]}$ taken for a normalization $\varepsilon_{23}\lambda_{[h]} = \varrho_2\phi_{[0]3} - \varrho_3\phi_{[0]2}$. This tensor is diagonal with respect to N–adapted (co) frames, $\mathbf{\Sigma}^{[\phi]\beta}_{\alpha} = \{\Upsilon_2, \Upsilon_2, 0, 0\}$ with

$$\Upsilon_2\left(x^2, x^3\right) = -\lambda^2_{[h]} \sin^2\left(\varrho_i x^i + \varphi_{[0]}\right). \tag{97}$$

So, we have the case from (135) and (136) with $\Upsilon_2(x^2, x^2, v) \to \Upsilon_2(x^2, x^2)$ and Υ_4, i. e.

$$G^2_2 = G^3_3 = -S^4_4 = \Upsilon_2\left(x^2, x^2\right) \text{ and } G^4_4 = G^4_4 = -R^2_2 = 0. \tag{98}$$

There are satisfied the compatibility conditions from Corollary 4.2. For the above stated ansatz for the d–metric and ϕ–field, the system (44) reduces to a particular case of (76)–(79), when the first equation is trivially satisfied by $g_2 = 1$ and $g_3 = -1$ but the second one is

$$S^4_4 = S^5_5 = -\frac{1}{2h_4h_5}\left[h_5^{**} - h_5^*\left(\ln\sqrt{|h_4h_5|}\right)^*]\right] = \lambda^2_{[h]} \sin^2\left(\varrho_i x^i + \varphi_{[0]}\right). \tag{99}$$

The right part of this equation is like a "cosmological constant", being nontrivial in the h–subspace and polarized by a nonmetricity and torsion wave (we can state $x^2 = t$ and choose the signature $(- + - - -)$).

The exact solution of (99) exists according the Theorem 4.3 (see formulas (83)–(86)). Taking any $h_4 = h_4[\lambda_{[h]} = 0]$ and $h_5 = h_5[\lambda_{[h]} = 0]$ solving the equation with $\lambda_{[h]} = 0$, for instance, like in (84), we can express the general solution with nontrivial source like

$$h_5[\lambda_{[h]}] = h_5,\ h_4[\lambda_{[h]}] = \varsigma_{[\lambda]}\left(x^i, v\right) h_4,$$

where (for an explicit source (97) in (86))

$$\varsigma_{[\lambda]}\left(t, x^3, v\right) = \varsigma_{4[0]}\left(t, x^3\right) - \frac{\lambda^2_{[h]}}{4}\sin^2\left(\varrho_2 t + \varrho_3 x^3 + \varphi_{[0]}\right)\int \frac{h_4h_5}{h_5^*}dv,$$

where $\varsigma_{4[0]}(t, x^3) = 1$ if we want to have $\varsigma_{[\lambda]}$ for $\lambda^2_{[h]} \to 0$. A particular class of 4D off–diagonal exact solutions with $h^*_{4,5} \neq 0$ (see the Corollary 4.3 with $x^2 = t$ stated to be the

time like coordinate and x^1 considered as the extra 5th dimensional one to be eliminated for reductions 5D→4D) is parametrized by the generic off–diagonal metric

$$\begin{aligned}
\delta s^2 &= (dt)^2 - \left(dx^3\right) - h_0^2(t,x^3)\left[f^*\left(t,x^3,v\right)\right]^2 |\varsigma_{[\lambda]}\left(t,x^3,v\right)|(\delta v)^2 - f^2\left(t,x^3,v\right)\left(\delta y^5\right)^2, \\
\delta v &= dv + w_{\widehat{k}}\left(t,x^3,v\right)dx^{\widehat{k}},\ \delta y^5 = dy^5 + n_{\widehat{k}}\left(t,x^3,v\right)dx^{\widehat{k}}, \qquad (100)
\end{aligned}$$

with coefficients of necessary smooth class, where $g_{\widehat{k}}\left(x^{\widehat{i}}\right)$ is a solution of the 2D equation (76) for a given source $\Upsilon_4\left(x^{\widehat{i}}\right)$,

$$\varsigma_{[\lambda]}\left(t,x^3,v\right) = 1 + \frac{\lambda_{[h]}^2}{16}h_0^2(t,x^3)\sin^2\left(\varrho_2 t + \varrho_3 x^3 + \varphi_{[0]}\right)f^2\left(t,x^3,v\right),$$

and the N–connection coefficients $N_{\widehat{i}}^4 = w_{\widehat{i}}(t,x^3,v)$ and $N_{\widehat{i}}^5 = n_{\widehat{i}}(t,x^3,v)$ are

$$w_{2,3} = -\frac{\partial_{2,3}\varsigma_{[\lambda]}\left(t,x^3,v\right)}{\varsigma_{[\lambda]}^*\left(t,x^3,v\right)}$$

and

$$n_{2,3}\left(t,x^3,v\right) = n_{2,3[1]}\left(t,x^3\right) + n_{2,3[2]}\left(t,x^3\right)\int\frac{\left[f^*\left(t,x^3,v\right)\right]^2}{\left[f\left(t,x^3,v\right)\right]^2}\varsigma_{[\lambda]}\left(t,x^3,v\right)dv,$$

define an exact 4D solution of the system of Einstein–Proca equations (46)–(49) for vanishing mass $\mu \to 0$, with holonomic and anholonomic variables and 1-form field

$$\phi_\mu = \left[\phi_{\widehat{i}} = \phi_{[0]\widehat{i}}\cos\left(\varrho_2 t + \varrho_3 x^3 + \varphi_{[0]}\right), \phi_4 = 0, \phi_0 = 0\right]$$

for arbitrary nontrivial functions $f\left(t,x^3,v\right)$ (with $f^* \neq 0$), $h_0^2(t,x^3)$, $n_{k[1,2]}\left(t,x^3\right)$ and sources $\Upsilon_2\left(t,x^3\right) = -\lambda_{[h]}^2\sin^2\left(\varrho_2 t + \varrho_3 x^3 + \varphi_{[0]}\right)$ and $\Upsilon_4 = 0$ and any integration constants to be defined by certain boundary conditions and additional physical arguments. For instance, we can consider ellipsoidal symmetries for the set of space coordinates $(x^3, y^4 = v, y^5)$ considered on possibility to be ellipsoidal ones, or even with topologically nontrivial configurations like torus, with toroidal coordinates. Such exact solutions emphasize anisotropic dependences on coordinate v and do not depend on y^5.

5.3 3D solitons in string Finsler–affine gravity

The d–metric (100) can be extended as to define a class of exact solutions of generalized Finsler affine string gravity (60), for certain particular cases describing 3D solitonic configurations.

We start with the the well known ansatz in string theory (see, for instance, [24]) for the H–field (57) when

$$\mathbf{H}_{\nu\lambda\rho} = \widehat{\mathbf{Z}}_{\ \nu\lambda\rho} + \widehat{\mathbf{H}}_{\nu\lambda\rho} = \lambda_{[H]}\sqrt{|\mathbf{g}_{\alpha\beta}|}\varepsilon_{\nu\lambda\rho} \qquad (101)$$

where $\varepsilon_{\nu\lambda\rho}$ is completely antisymmetric and $\lambda_{[H]} = const$, which satisfies the field equations for $\mathbf{H}_{\nu\lambda\rho}$, see (59). The ansatz (101) is chosen for a locally anisotropic background with

$\widehat{\mathbf{Z}}_{\ \nu\lambda\rho}$ defined by the d–torsions for the canonical d–connection. So, the values $\widehat{\mathbf{H}}_{\nu\lambda\rho}$ are constrained to solve the equations (101) for a fixed value of the cosmological constant $\lambda_{[H]}$ effectively modeling some corrections from string gravity. In this case, the source (97) is modified to

$$\mathbf{\Sigma}_{\alpha}^{[\phi]\beta}+\mathbf{\Sigma}_{\alpha}^{[\mathbf{H}]\beta}=\{\Upsilon_2+\frac{\lambda_{[H]}^2}{4},\Upsilon_2+\frac{\lambda_{[H]}^2}{4},\frac{\lambda_{[H]}^2}{4},\frac{\lambda_{[H]}^2}{4}\}$$

and the equations (98) became more general,

$$G_2^2=G_3^3=-S_4^4=\Upsilon_2\left(x^2,x^2\right)+\frac{\lambda_{[H]}^2}{4}\text{ and }G_4^4=G_4^4=-R_2^2=\frac{\lambda_{[H]}^2}{4}, \tag{102}$$

or, in component form

$$R_2^2 = R_3^3=-\frac{1}{2g_2g_3}[g_3^{\bullet\bullet}-\frac{g_2^{\bullet}g_3^{\bullet}}{2g_2}-\frac{(g_3^{\bullet})^2}{2g_3}+g_2^{''}-\frac{g_2^{'}g_3^{'}}{2g_3}-\frac{(g_2^{'})^2}{2g_2}]=-\frac{\lambda_{[H]}^2}{4}, \tag{103}$$

$$S_4^4 = S_5^5=-\frac{1}{2h_4h_5}\left[h_5^{**}-h_5^{*}\left(\ln\sqrt{|h_4h_5|}\right)^{*}]\right]=-\frac{\lambda_{[H]}^2}{4}+\lambda_{[h]}^2\sin^2\left(\varrho_i x^i+\varphi_{[0]}\right). \tag{104}$$

The solution of (103) can be found as in the case for (82), when $\psi=\ln|g_2|=\ln|g_3|$ is a solution of

$$\ddot{\psi}+\psi''=-\frac{\lambda_{[H]}^2}{2}, \tag{105}$$

where, for simplicity we choose the h–variables $x^2=\tilde{x}^2$ and $x^3=\tilde{x}^3$.

The solution of (104) can be constructed similarly to the equation (99) but for a modified source (see Theorem 4.3 and formulas (83)–(86)). Taking any $h_4=h_4[\lambda_{[h]}=0,\lambda_{[H]}=0]$ and $h_5=h_5[\lambda_{[h]}=0,\lambda_{[H]}=0]$ solving the equation with $\lambda_{[h]}=0$ and $\lambda_{[H]}=0$ like in (84), we can express the general solution with nontrivial source like

$$h_5[\lambda_{[h]},\lambda_{[H]}]=h_5,\ h_4[\lambda_{[h]},\lambda_{[H]}]=\varsigma_{[\lambda,H]}\left(x^i,v\right)h_4,$$

where (for an explicit source from (104) in (86))

$$\varsigma_{[\lambda,H]}\left(t,x^3,v\right)=\varsigma_{4[0]}\left(t,x^3\right)-\frac{1}{4}\left[\lambda_{[h]}^2\sin^2\left(\varrho_2 t+\varrho_3x^3+\varphi_{[0]}\right)-\frac{\lambda_{[H]}^2}{4}\right]\int\frac{h_4h_5}{h_5^*}dv,$$

where $\varsigma_{4[0]}\left(t,x^3\right)=1$ if we want to have $\varsigma_{[\lambda]}$ for $\lambda_{[h]}^2,\lambda_{[H]}^2\to 0$.

We define a class of 4D off–diagonal exact solutions of the system (60) with $h_{4,5}^*\neq 0$ (see the Corollary 4.3 with $x^2=t$ stated to be the time like coordinate and x^1 considered as the extra 5th dimensional one to be eliminated for reductions 5D→4D) is parametrized by the generic off–diagonal metric

$$\begin{aligned}\delta s^2 &= e^{\psi(t,x^3)}(dt)^2-e^{\psi(t,x^3)}\left(dx^3\right)-f^2\left(t,x^3,v\right)\left(\delta y^5\right)^2\\ &\quad -h_0^2(t,x^3)\left[f^*\left(t,x^3,v\right)\right]^2|\varsigma_{[\lambda,H]}\left(t,x^3,v\right)|\left(\delta v\right)^2, \\ \delta v &= dv+w_{\widehat{k}}\left(t,x^3,v\right)dx^{\widehat{k}},\ \delta y^5=dy^5+n_{\widehat{k}}\left(t,x^3,v\right)dx^{\widehat{k}},\end{aligned} \tag{106}$$

with coefficients of necessary smooth class, where $g_{\widehat{k}}\left(x^{\widehat{i}}\right)$ is a solution of the 2D equation (76) for a given source $\Upsilon_4\left(x^{\widehat{i}}\right)$,

$$\varsigma_{[\lambda,H]}\left(t,x^3,v\right)=1+\frac{h_0^2(t,x^3)}{16}\left[\lambda_{[h]}^2\sin^2\left(\varrho_2 t+\varrho_3 x^3+\varphi_{[0]}\right)-\frac{\lambda_{[H]}^2}{4}\right]f^2\left(t,x^3,v\right),$$

and the N–connection coefficients $N_{\widehat{i}}^4=w_{\widehat{i}}(t,x^3,v)$ and $N_{\widehat{i}}^5=n_{\widehat{i}}(t,x^3,v)$ are

$$w_{2,3}=-\frac{\partial_{2,3}\varsigma_{[\lambda,H]}\left(t,x^3,v\right)}{\varsigma_{[\lambda,H]}^*\left(t,x^3,v\right)}$$

and

$$n_{2,3}\left(t,x^3,v\right)=n_{2,3[1]}\left(t,x^3\right)+n_{2,3[2]}\left(t,x^3\right)\int\frac{\left[f^*\left(t,x^3,v\right)\right]^2}{\left[f\left(t,x^3,v\right)\right]^2}\varsigma_{[\lambda,H]}\left(t,x^3,v\right)dv,$$

define an exact 4D solution of the system of generalized Finsler–affine gravity equations (60) for vanishing Proca mass $\mu\rightarrow 0$, with holonomic and anholonomic variables, 1-form field

$$\phi_\mu=\left[\phi_{\widehat{i}}=\phi_{[0]\widehat{i}}(t,x^3)\cos\left(\varrho_2 t+\varrho_3 x^3+\varphi_{[0]}\right),\phi_4=0,\phi_0=0\right] \tag{107}$$

and nontrivial effective H–field $\mathbf{H}_{\nu\lambda\rho}=\lambda_{[H]}\sqrt{|\mathbf{g}_{\alpha\beta}|}\varepsilon_{\nu\lambda\rho}$ for arbitrary nontrivial functions $f\left(t,x^3,v\right)$ (with $f^*\neq 0$), $h_0^2(t,x^3)$, $n_{k[1,2]}\left(t,x^3\right)$ and sources

$$\Upsilon_2\left(t,x^3\right)=\lambda_{[H]}^2/4-\lambda_{[h]}^2(t,x^3)\sin^2\left(\varrho_2 t+\varrho_3 x^3+\varphi_{[0]}\right)\text{ and }\Upsilon_4=\lambda_{[H]}^2/4$$

and any integration constants to be defined by certain boundary conditions and additional physical arguments. The function $\phi_{[0]\widehat{i}}(t,x^3)$ in (107) is taken to solve the equation

$$\partial_2[e^{-\psi\left(t,x^3\right)}\partial_2\phi_k]-\partial_3[e^{-\psi\left(t,x^3\right)}\partial_3\phi_k]=L_{ki}^j\partial^i\phi_j-L_{ij}^i\partial^j\phi_k \tag{108}$$

where L_{ki}^j are computed for the d–metric (106) following the formulas (27). For $\psi=0$, we obtain just the plane wave equation (95) when $\phi_{[0]\widehat{i}}$ and $\lambda_{[h]}^2(t,x^3)$ reduce to constant values. We do not fix here any value of $\psi\left(t,x^3\right)$ solving (105) in order to define explicitly a particular solution of (108). We note that for any value of $\psi\left(t,x^3\right)$ we can solve the inhomogeneous wave equation (108) by using solutions of the homogeneous case.

For simplicity, we do not present here the explicit value of $\sqrt{|\mathbf{g}_{\alpha\beta}|}$ computed for the d–metric (106) as well the values for distorsions $\widehat{\mathbf{Z}}_{\ \nu\lambda\rho}$, defined by d–torsions of the canonical d–connection, see formulas (53) and (29) (the formulas are very cumbersome and do not reflect additional physical properties). Having defined $\widehat{\mathbf{Z}}_{\ \nu\lambda\rho}$, we can compute

$$\widehat{\mathbf{H}}_{\nu\lambda\rho}=\lambda_{[H]}\sqrt{|\mathbf{g}_{\alpha\beta}|}\varepsilon_{\nu\lambda\rho}-\widehat{\mathbf{Z}}_{\ \nu\lambda\rho}.$$

We note that the torsion $\widehat{\mathbf{T}}^\nu{}_{\lambda\rho}$ contained in $\widehat{\mathbf{Z}}_{\ \nu\lambda\rho}$, related to string corrections by the H–field, is different from the torsion $\mathbf{T}=k_2\phi$ and nontrivial nonmetricity $\mathbf{Q}=k_0\phi$, $\mathbf{\Lambda}=k_1\phi$, from the metric–affine part of the theory, see (42).

We can choose the function $f(t, x^3, v)$ from (106), or (100), as it would be a solution of the Kadomtsev–Petviashvili (KdP) equation [25], i. e. to satisfy

$$f^{\bullet\bullet} + \epsilon \left(f' + 6ff^* + f^{***}\right)^* = 0, \ \epsilon = \pm 1,$$

or, for another locally anisotropic background, to satisfy the (2 + 1)–dimensional sine–Gordon (SG) equation,

$$-f^{\bullet\bullet} + f' + f^{**} = \sin f,$$

see Refs. [26] on gravitational solitons and theory of solitons. In this case, we define a nonlinear model of graviational plane wave and 3D solitons in the framework of the MAG with string corrections by H–field. Such solutions generalized those considered in Refs. [7] for 4D and 5D gravity.

We can also consider that $F/L = f^2(t, x^3, v)$ is just the generation function for a 2D model of Finsler/Lagrange geometry (being of any solitonic or another type nature). In this case, the geometric background is characterized by this type locally anisotropic configurations (for Finsler metrics we shall impose corresponding homogeneity conditions on coordinates).

6 Final Remarks

In this paper we have investigated the dynamical aspects of metric–affine gravity (MAG) with certain additional string corrections defined by the antisymmetric H–field when the metric structure is generic off–diagonal and the spacetime is provided with an anholonomic frame structure with associated nonlinear connection (N–connection). We analyzed the corresponding class of Lagrangians and derived the field equations of MAG and string gravity with mixed holonomic and anholonomic variables. The main motivation for this work is to determine the place and significance of such models of gravity which contain as exact solutions certain classes of metrics and connections modeling Finsler like geometries even in the limits to the general relativity theory.

The work supports the results of Refs. [7, 8] where various classes of exact solutions in Einstein, Einstein–Cartan, gauge and string gravity modeling Finsler–Lagrange configurations were constructed. We provide an irreducible decomposition techniques (in our case with additional N–connection splitting) and study the dynamics of MAG fields generating the locally anisotropic geometries and interactions classified in Ref. [6]. There are proved the main theorems on irreducible reduction to effective Einstein–Proca equations with string corrections and formulated a new method of constructing exact solutions.

As explicit examples of the new type of locally anisotropic configurations in MAG and string gravity, we have elaborated three new classes of exact solutions depending on 3-4 variables possessing nontrivial torsion and nonmetricity fields, describing plane wave and three dimensional soliton interactions and induced generalized Finsler–affine effective configurations.

Finally, it seems worthwhile to note that such Finsler like configurations do not violates the postulates of the general relativity theory in the corresponding limits to the four dimensional Einstein theory because such metrics transform into exact solutions of this theory. The anisotropies are modeled by certain anholonomic frame constraints on a (pseudo)

Riemannian spacetime. In this case the restrictions imposed on physical applications of the Finsler geometry, derived from experimental data on possible limits for brocken local Lorentz invariance (see, for instance, Ref. [27]), do not hold.

A Proof of Theorem 4.1

We give some details on straightforward calculations outlined in Ref. [15] for (pseudo) Riemannian and Riemann–Cartan spaces. In brief, the proof of Theorem 4.1 is to be performed in this way: Introducing $N_i^4 = w_i$ and $N_i^5 = n_i$ in (6) and (7) and re–writting (63) into a diagonal (in our case) block form (11), we compute the h- and v–irreducible components of the cannonical d–connection (27). The next step is to compute d–curvatures (30) and by contracting of indices to define the components of the Ricci d–tensor (31) which results in (64)–(67). We emphasize that such computations can not be performed directly by applying any Tensor, Maple of Mathematica macros because, in our case, we consider canonical d–connections instead of the Levi–Civita connection [23]. We give the details of such calculus related to N–adapted anholonomic frames.

The five dimensional (5D) local coordinates are x^i and $y^a = (v, y)$, i. e. $y^4 = v$, $y^5 = y$, were indices $i, j, k... = 1, 2, 3$ and $a, b, c, ... = 4, 5$. Our reductions to 4D will be considered by excluding dependencies on the variable x^1 and for trivial embeddings of 4D off–diagonal ansatz into 5D ones. The signatures of metrics coud be arbitrary ones. In general, the spacetime could be with torsion, but we shall always be interested to define the limits to (pseudo) Riemannian spaces.

The d–metric (11) for an ansatz (63) with $g_1 = const$, is written

$$\begin{aligned} \delta s^2 &= g_1(dx^1)^2 + g_2\left((x^2, x^3\right)(dx^2)^2 + g_3\left(x^k\right)(dx^3)^2 + h_4\left(x^k, v\right)(\delta v)^2 + h_5\left(x^k, v\right)(\delta y)^2, \\ \delta v &= dv + w_i\left(x^k, v\right)dx^i,\ \delta y = dy + n_i\left(x^k, v\right)dx^i \end{aligned} \tag{109}$$

when the generic off–diagonal metric (62) is associated to a N–connection structure N_i^a with $N_i^4 = w_i\left(x^k, v\right)$ and $N_i^5 = n_i\left(x^k, v\right)$. We note that the metric (109) does not depend on variable $y^5 = y$, but emphasize the dependence on "anisotropic" variable $y^4 = v$.

If we regroup (109) with respect to true differentials $du^\alpha = (dx^i, dy^a)$ we obtain just the ansatz (63). It is a cumbersome task to perform tensor calculations (for instance, of curvature and Ricci tensors) with such generic off–diagonal ansatz but the formulas simplify substantially with respect to N–adapted frames of type(6) and (7) and for effectively diagonalized metrics like (109).

So, the metric (62) transform in a diagonal one with respect to the pentads (frames, funfbeins)

$$e^i = dx^i, e^4 = \delta v = dv + w_i\left(x^k, v\right)dx^i, e^5 = \delta y = dy + n_i\left(x^k, v\right)dx^i \tag{110}$$

or

$$\delta u^\alpha = \left(dx^i, \delta y^a = dy^a + N_i^a dx^i\right)$$

being dual to the N–elongated partial derivative operators,

$$
\begin{aligned}
e_1 &= \delta_1 = \frac{\partial}{\partial x^1} - N_1^a \frac{\partial}{\partial y^a} = \frac{\partial}{\partial x^1} - w_1 \frac{\partial}{\partial v} - n_1 \frac{\partial}{\partial y}, \\
e_2 &= \delta_2 = \frac{\partial}{\partial x^2} - N_2^a \frac{\partial}{\partial y^a} = \frac{\partial}{\partial x^2} - w_2 \frac{\partial}{\partial v} - n_2 \frac{\partial}{\partial y}, \\
e_3 &= \delta_3 = \frac{\partial}{\partial x^3} - N_3^a \frac{\partial}{\partial y^a} = \frac{\partial}{\partial x^3} - w_3 \frac{\partial}{\partial v} - n_3 \frac{\partial}{\partial y}, \\
e_4 &= \frac{\partial}{\partial y^4} = \frac{\partial}{\partial v},\ e_5 = \frac{\partial}{\partial y^5} = \frac{\partial}{\partial y}
\end{aligned}
\tag{111}
$$

when $\delta_\alpha = \frac{\delta}{\partial u^\alpha} = \left(\frac{\delta}{\partial x^i} = \frac{\partial}{\partial x^i} - N_i^a \frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b} \right).$

The N–elongated partial derivatives of a function $f(u^\alpha) = f(x^i, y^a) = f(x, r, v, y)$ are computed in the formSo the N–elongated derivatives are

$$
\delta_2 f = \frac{\delta f}{\partial u^2} = \frac{\delta f}{\partial x^2} = \frac{\delta f}{\partial x} = \frac{\partial f}{\partial x} - N_2^a \frac{\partial f}{\partial y^a} = \frac{\partial f}{\partial x} - w_2 \frac{\partial f}{\partial v} - n_2 \frac{\partial f}{\partial y} = f^\bullet - w_2\ f' - n_2\ f^*
$$

where

$$
f^\bullet = \frac{\partial f}{\partial x^2} = \frac{\partial f}{\partial x},\ f' = \frac{\partial f}{\partial x^3} = \frac{\partial f}{\partial r},\ f^* = \frac{\partial f}{\partial y^4} = \frac{\partial f}{\partial v}.
$$

The N–elongated differential is

$$
\delta f = \frac{\delta f}{\partial u^\alpha} \delta u^\alpha.
$$

The N–elongated differential calculus should be applied if we work with respect to N–adapted frames.

A.1 Calculation of N–connection curvature

We compute the coefficients (5) for the d–metric (109) (equivalently, the ansatz (63)) defining the curvature of N–connection N_i^a, by substituting $N_i^4 = w_i\left(x^k, v\right)$ and $N_i^5 = n_i\left(x^k, v\right),$ where $i = 2, 3$ and $a = 4, 5$. The result for nontrivial values is

$$
\begin{aligned}
\Omega_{23}^4 &= -\Omega_{23}^4 = w_2' - w_3^{\cdot} - w_3 w_2^* - w_2 w_3^*, \\
\Omega_{23}^5 &= -\Omega_{23}^5 = n_2' - n_3^{\cdot} - w_3 n_2^* - w_2 n_3^*.
\end{aligned}
\tag{112}
$$

The canonical d–connection $\widehat{\mathbf{\Gamma}}^\gamma{}_{\alpha\beta} = \left(\widehat{L}^i_{jk}, \widehat{L}^a_{bk}, \widehat{C}^i_{jc}, \widehat{C}^a_{bc} \right)$ (27) defines the covariant derivative $\widehat{\mathbf{D}}$, satisfying the metricity conditions $\widehat{\mathbf{D}}_\alpha \mathbf{g}_{\gamma\delta} = 0$ for $\mathbf{g}_{\gamma\delta}$ being the metric (109) with the coefficients written with respect to N–adapted frames. $\widehat{\mathbf{\Gamma}}^\gamma{}_{\alpha\beta}$ has nontrivial d–torsions.

We compute the Einstein tensors for the canonica d–connection $\widehat{\mathbf{\Gamma}}^\gamma{}_{\alpha\beta}$ defined by the ansatz (109) with respect to N–adapted frames (110) and (111). This results in exactly integrable vacuum Einstein equations and certain type of sources. Such solutions could be with nontrivial torsion for different classes of linear connections from Riemann–Cartan and generalized Finsler geometries. So, the anholonomic frame method offers certain possibilites to be extended to in string gravity where the torsion could be not zero. But we can always select the limit to Levi–Civita connections, i. e. to (pseudo) Riemannian spaces by considering additional constraints, see Corollary 3.3 and/or conditions (61).

A.2 Calculation of the canonical d–connection

We compute the coefficients (27) for the d–metric (109) (equivalently, the ansatz (63)) when $g_{jk} = \{g_j\}$ and $h_{bc} = \{h_b\}$ are diagonal and g_{ik} depend only on x^2 and x^3 but not on y^a.

We have

$$\begin{aligned} \delta_k g_{ij} &= \partial_k g_{ij} - w_k g^*_{ij} = \partial_k g_{ij},\ \delta_k h_b = \partial_k h_b - w_k h^*_b \\ \delta_k w_i &= \partial_k w_i - w_k w^*_i,\ \delta_k n_i = \partial_k n_i - w_k n^*_i \end{aligned} \tag{113}$$

resulting in formulas

$$\widehat{L}^i_{jk} = \frac{1}{2} g^{ir} \left(\frac{\delta g_{jk}}{\delta x^k} + \frac{\delta g_{kr}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^r} \right) = \frac{1}{2} g^{ir} \left(\frac{\partial g_{jk}}{\delta x^k} + \frac{\partial g_{kr}}{\delta x^j} - \frac{\partial g_{jk}}{\delta x^r} \right)$$

The nontrivial values of $\widehat{L}^i_{jk}$ are

$$\begin{aligned} \widehat{L}^2_{22} &= \frac{g^\bullet_2}{2g_2} = \alpha^\bullet_2,\ \widehat{L}^2_{23} = \frac{g'_2}{2g_2} = \alpha'_2,\ \widehat{L}^2_{33} = -\frac{g^\bullet_3}{2g_2} \\ \widehat{L}^3_{22} &= -\frac{g'_2}{2g_3},\ \widehat{L}^3_{23} = \frac{g^\bullet_3}{2g_3} = \alpha^\bullet_3,\quad \widehat{L}^3_{33} = \frac{g'_3}{2g_3} = \alpha'_3. \end{aligned} \tag{114}$$

In a similar form we compute the components

$$\widehat{L}^a_{bk} = \partial_b N^a_k + \frac{1}{2} h^{ac} \left(\partial_k h_{bc} - N^d_k \frac{\partial h_{bc}}{\partial y^d} - h_{dc} \partial_b N^d_k - h_{db} \partial_c N^d_k \right)$$

having nontrivial values

$$\begin{aligned} \widehat{L}^4_{42} &= \frac{1}{2h_4} \left(h^\bullet_4 - w_2 h^*_4 \right) = \delta_2 \ln \sqrt{|h_4|} \doteqdot \delta_2 \beta_4, \\ \widehat{L}^4_{43} &= \frac{1}{2h_4} \left(h'_4 - w_3 h^*_4 \right) = \delta_3 \ln \sqrt{|h_4|} \doteqdot \delta_3 \beta_4 \end{aligned} \tag{115}$$

$$\widehat{L}^4_{5k} = -\frac{h_5}{2h_4} n^*_k,\ \widehat{L}^5_{bk} = \partial_b n_k + \frac{1}{2h_5} \left(\partial_k h_{b5} - w_k h^*_{b5} - h_5 \partial_b n_k \right), \tag{116}$$

$$\begin{aligned} \widehat{L}^5_{4k} &= n^*_k + \frac{1}{2h_5} \left(-h_5 n^*_k \right) = \frac{1}{2} n^*_k, \\ \widehat{L}^5_{5k} &= \frac{1}{2h_5} \left(\partial_k h_5 - w_k h^*_5 \right) = \delta_k \ln \sqrt{|h_4|} = \delta_k \beta_4. \end{aligned} \tag{117}$$

We note that

$$\widehat{C}^i_{jc} = \frac{1}{2} g^{ik} \frac{\partial g_{jk}}{\partial y^c} \doteqdot 0 \tag{118}$$

because $g_{jk} = g_{jk}(x^i)$ for the considered ansatz.

The values

$$\widehat{C}^a_{bc} = \frac{1}{2} h^{ad} \left(\frac{\partial h_{bd}}{\partial y^c} + \frac{\partial h_{cd}}{\partial y^b} - \frac{\partial h_{bc}}{\partial y^d} \right)$$

for $h_{bd} = [h_4, h_5]$ from the ansatz (63) have nontrivial components

$$\widehat{C}^4_{44} = \frac{h_4^*}{2h_4} \doteqdot \beta_4^*, \widehat{C}^4_{55} = -\frac{h_5^*}{2h_4}, \widehat{C}^5_{45} = \frac{h_5^*}{2h_5} \doteqdot \beta_5^*. \tag{119}$$

The set of formulas (114)–(119) define the nontrivial coefficients of the canonical d–connection $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = \left(\widehat{L}^i_{jk}, \widehat{L}^a_{bk}, \widehat{C}^i_{jc}, \widehat{C}^a_{bc}\right)$ (27) for the 5D ansatz (109).

A.3 Calculation of torsion coefficients

We should put the nontrivial values (114)– (119) into the formulas for d–torsion (29).

One holds $T^i_{.jk} = 0$ and $T^a_{.bc} = 0$, because of symmetry of coefficients L^i_{jk} and C^a_{bc}.

We have computed the nontrivial values of $\Omega^a_{.ji}$, see (112) resulting in

$$\begin{aligned} T^4_{23} &= \Omega^4_{23} = -\Omega^4_{23} = w_2' - w_3^\bullet - w_3 w_2^* - w_2 w_3^*, \\ T^5_{23} &= \Omega^5_{23} = -\Omega^5_{23} = n_2' - n_3^\bullet - w_3 n_2^* - w_2 n_3^*. \end{aligned} \tag{120}$$

One follows

$$T^i_{ja} = -T^i_{aj} = C^i_{.ja} = \widehat{C}^i_{jc} = \frac{1}{2} g^{ik} \frac{\partial g_{jk}}{\partial y^c} \doteqdot 0,$$

see (118).

For the components

$$T^a_{.bi} = -T^a_{.ib} = P^a_{.bi} = \frac{\partial N^a_i}{\partial y^b} - L^a_{.bj},$$

i. e. for

$$\widehat{P}^4_{.bi} = \frac{\partial N^4_i}{\partial y^b} - \widehat{L}^4_{.bj} = \partial_b w_i - \widehat{L}^4_{.bj} \text{ and } \widehat{P}^5_{.bi} = \frac{\partial N^5_i}{\partial y^b} - \widehat{L}^5_{.bj} = \partial_b n_i - \widehat{L}^5_{.bj},$$

we have the nontrivial values

$$\begin{aligned} \widehat{P}^4_{.4i} &= w_i^* - \frac{1}{2h_4}\left(\partial_i h_4 - w_i h_4^*\right) = w_i^* - \delta_i \beta_4, \ \widehat{P}^4_{.5i} = \frac{h_5}{2h_4} n_i^*, \\ \widehat{P}^5_{.4i} &= \frac{1}{2} n_i^*, \ \widehat{P}^5_{.5i} = -\frac{1}{2h_5}\left(\partial_i h_5 - w_i h_5^*\right) = -\delta_i \beta_5. \end{aligned} \tag{121}$$

The formulas (120) and (121) state the nontrivial coefficients of the canonical d–connection for the chosen ansatz (109).

A.4 Calculation of the Ricci tensor

Let us compute the value $R_{ij} = R^k{}_{ijk}$ as in (31) for

$$R^i{}_{hjk} = \frac{\delta L^i_{.hj}}{\delta x^k} - \frac{\delta L^i_{.hk}}{\delta x^j} + L^m_{.hj} L^i_{mk} - L^m_{.hk} L^i_{mj} - C^i_{.ha} \Omega^a_{.jk},$$

from (30). It should be noted that $C^i_{.ha} = 0$ for the ansatz under consideration, see (118). We compute

$$\frac{\delta L^i_{.hj}}{\delta x^k} = \partial_k L^i_{.hj} + N^a_k \partial_a L^i_{.hj} = \partial_k L^i_{.hj} + w_k \left(L^i_{.hj}\right)^* = \partial_k L^i_{.hj}$$

because $L^i_{.hj}$ do not depend on variable $y^4 = v$.

Derivating (114), we obtain

$$\begin{aligned}
\partial_2 L^2{}_{22} &= \frac{\ddot{g}_2}{2g_2} - \frac{(\dot{g}_2)^2}{2\,(g_2)^2},\ \partial_2 L^2{}_{23} = \frac{\dot{g}_2'}{2g_2} - \frac{\dot{g}_2 g_2'}{2\,(g_2)^2},\ \partial_2 L^2{}_{33} = -\frac{\ddot{g}_3}{2g_2} + \frac{\dot{g}_2 \dot{g}_3}{2\,(g_2)^2}, \\
\partial_2 L^3{}_{22} &= -\frac{\dot{g}_2'}{2g_3} + \frac{\dot{g}_2 g_3'}{2\,(g_3)^2}, \partial_2 L^3{}_{23} = \frac{\ddot{g}_3}{2g_3} - \frac{(\dot{g}_3)^2}{2\,(g_3)^2},\ \partial_2 L^3{}_{33} = \frac{\dot{g}_3'}{2g_3} - \frac{\dot{g}_3 g_3'}{2\,(g_3)^2}, \\
\partial_3 L^2{}_{22} &= \frac{\dot{g}_2'}{2g_2} - \frac{\dot{g}_2 g_2'}{2\,(g_2)^2}, \partial_3 L^2{}_{23} = \frac{g_2^{ll}}{2g_2} - \frac{\left(g_2^l\right)^2}{2\,(g_2)^2}, \partial_3 L^2{}_{33} = -\frac{\dot{g}_3'}{2g_2} + \frac{\dot{g}_3 g_2'}{2\,(g_2)^2}, \\
\partial_3 L^3{}_{22} &= -\frac{g_2''}{2g_3} + \frac{\dot{g}_2 g_2'}{2\,(g_3)^2}, \partial_3 L^3{}_{23} = \frac{\dot{g}_3'}{2g_3} - \frac{\dot{g}_3 g_3'}{2\,(g_3)^2}, \partial_3 L^3{}_{33} = \frac{g_3^{ll}}{2g_3} - \frac{\left(g_3^l\right)^2}{2\,(g_3)^2}.
\end{aligned}$$

For these values and (114), there are only 2 nontrivial components,

$$\begin{aligned}
R^2{}_{323} &= \frac{\ddot{g}_3}{2g_2} - \frac{\dot{g}_2 \dot{g}_3}{4\,(g_2)^2} - \frac{(\dot{g}_3)^2}{4g_2g_3} + \frac{g_2^{ll}}{2g_2} - \frac{g_2^l g_3^l}{4g_2g_3} - \frac{\left(g_2^l\right)^2}{4\,(g_2)^2} \\
R^3{}_{223} &= -\frac{\ddot{g}_3}{2g_3} + \frac{\dot{g}_2 \dot{g}_3}{4g_2g_3} + \frac{(\dot{g}_3)^2}{4(g_3)^2} - \frac{g_2^{ll}}{2g_3} + \frac{g_2^l g_3^l}{4(g_3)^2} + \frac{\left(g_2^l\right)^2}{4g_2g_3}
\end{aligned}$$

with

$$R_{22} = -R^3{}_{223} \text{ and } R_{33} = R^2{}_{323},$$

or

$$R^2_2 = R^3_3 = -\frac{1}{2g_2g_3}\left[\ddot{g}_3 - \frac{\dot{g}_2\dot{g}_3}{2g_2} - \frac{(\dot{g}_3)^2}{2g_3} + g_2'' - \frac{g_2^l g_3^l}{2g_3} - \frac{\left(g_2^l\right)^2}{2g_2}\right]$$

which is (64).

Now, we consider

$$\begin{aligned}
P^c{}_{bka} &= \frac{\partial L^c_{.bk}}{\partial y^a} - \left(\frac{\partial C^c_{.ba}}{\partial x^k} + L^c_{.dk} C^d_{.ba} - L^d_{.bk} C^c_{.da} - L^d_{.ak} C^c_{.bd}\right) + C^c_{.bd} P^d_{.ka} \\
&= \frac{\partial L^c_{.bk}}{\partial y^a} - C^c_{.ba|k} + C^c_{.bd} P^d_{.ka}
\end{aligned}$$

from (30). Contracting indices, we have

$$R_{bk} = P^a{}_{bka} = \frac{\partial L^a_{.bk}}{\partial y^a} - C^a_{.ba|k} + C^a_{.bd} P^d_{.ka}$$

Let us denote $C_b = C^c_{.ba}$ and write

$$C_{.b|k} = \delta_k C_b - L^d{}_{bk} C_d = \partial_k C_b - N^e_k \partial_e C_b - L^d{}_{bk} C_d = \partial_k C_b - w_k C_b^* - L^d{}_{bk} C_d.$$

We express

$$R_{bk} = {}_{[1]}R_{bk} + {}_{[2]}R_{bk} + {}_{[3]}R_{bk}$$

where

$$\begin{aligned}
{}_{[1]}R_{bk} &= \left(L_{bk}^{4}\right)^{*}, \\
{}_{[2]}R_{bk} &= -\partial_k C_b + w_k C_b^{*} + L^{d}{}_{bk} C_d, \\
{}_{[3]}R_{bk} &= C_{.bd}^{a} P_{.ka}^{d} = C_{.b4}^{4} P_{.k4}^{4} + C_{.b5}^{4} P_{.k4}^{5} + C_{.b4}^{5} P_{.k5}^{4} + C_{.b5}^{5} P_{.k5}^{5}
\end{aligned}$$

and

$$\begin{aligned}
C_4 &= C_{44}^{4} + C_{45}^{5} = \frac{h_4^{*}}{2h_4} + \frac{h_5^{*}}{2h_5} = \beta_4^{*} + \beta_5^{*}, \\
C_5 &= C_{54}^{4} + C_{55}^{5} = 0
\end{aligned} \tag{122}$$

see(119) .

We compute

$$R_{4k} = {}_{[1]}R_{4k} + {}_{[2]}R_{4k} + {}_{[3]}R_{4k}$$

with

$$\begin{aligned}
{}_{[1]}R_{4k} &= \left(L_{4k}^{4}\right)^{*} = \left(\delta_k \beta_4\right)^{*} \\
{}_{[2]}R_{4k} &= -\partial_k C_4 + w_k C_4^{*} + L^{4}{}_{4k} C_4, L^{4}{}_{4k} = \delta_k \beta_4 \text{ see (115} \\
&= -\partial_k \left(\beta_4^{*} + \beta_5^{*}\right) + w_k \left(\beta_4^{*} + \beta_5^{*}\right)^{*} + L^{4}{}_{4k} \left(\beta_4^{*} + \beta_5^{*}\right) \\
{}_{[3]}R_{4k} &= C_{.44}^{4} P_{.k4}^{4} + C_{.45}^{4} P_{.k4}^{5} + C_{.44}^{5} P_{.k5}^{4} + C_{.45}^{5} P_{.k5}^{5} \\
&= \beta_4^{*} \left(w_k^{*} - \delta_k \beta_4\right) - \beta_5^{*} \delta_k \beta_5
\end{aligned}$$

Summarizing, we get

$$R_{4k} = w_k \left[\beta_5^{**} + \left(\beta_5^{*}\right)^2 - \beta_4^{*} \beta_5^{*}\right] + \beta_5^{*} \partial_k \left(\beta_4 + \beta_5\right) - \partial_k \beta_5^{*}$$

or, for

$$\beta_4^{*} = \frac{h_4^{*}}{2h_4}, \partial_k \beta_4 = \frac{\partial_k h_4}{2h_4}, \beta_5^{*} = \frac{h_5^{*}}{2h_5}, \beta_5^{**} = \frac{h_5^{**} h_5 - \left(h_5^{*}\right)^2}{2\left(h_5\right)^5},$$

we can write

$$2h_5 R_{4k} = w_k \left[h_5^{**} - \frac{\left(h_5^{*}\right)^2}{2h_5} - \frac{h_4^{*} h_5^{*}}{2h_4}\right] + \frac{h_5^{*}}{2} \left(\frac{\partial_k h_4}{h_4} + \frac{\partial_k h_5}{h_5}\right) - \partial_k h_5^{*}$$

which is equivalent to (66)

In a similar way, we compute

$$R_{5k} = {}_{[1]}R_{5k} + {}_{[2]}R_{5k} + {}_{[3]}R_{5k}$$

with

$$\begin{aligned}
{}_{[1]}R_{5k} &= \left(L_{5k}^{4}\right)^{*}, \\
{}_{[2]}R_{5k} &= -\partial_k C_5 + w_k C_5^{*} + L^{4}{}_{5k} C_4, \\
{}_{[3]}R_{5k} &= C_{.54}^{4} P_{.k4}^{4} + C_{.55}^{4} P_{.k4}^{5} + C_{.54}^{5} P_{.k5}^{4} + C_{.55}^{5} P_{.k5}^{5}.
\end{aligned}$$

We have

$$
\begin{aligned}
R_{5k} &= \left(L^4_{5k}\right)^* + L^4{}_{5k}C_4 + C^4_{.55}P^5_{.k4} + C^5_{.54}P^4_{.k5} \\
&= \left(-\frac{h_5}{h_4}n^*_k\right)^* - \frac{h_5}{h_4}n^*_k\left(\frac{h^*_4}{2h_4} + \frac{h^*_5}{2h_5}\right) + \frac{h^*_5}{2h_5}\frac{h_5}{2h_4}n^*_k - \frac{h^*_5}{2h_4}\frac{1}{2}n^*_k
\end{aligned}
$$

which can be written

$$
2h_4R_{5k} = h_5n^{**}_k + \left(\frac{h_5}{h_4}h^*_4 - \frac{3}{2}h^*_5\right)n^*_k
$$

i. e. (67)

For the values

$$
P^i{}_{jka} = \frac{\partial L^i_{.jk}}{\partial y^k} - \left(\frac{\partial C^i_{.ja}}{\partial x^k} + L^i_{.lk}C^l_{.ja} - L^l_{.jk}C^i_{.la} - L^c_{.ak}C^i_{.jc}\right) + C^i_{.jb}P^b_{.ka}
$$

from (30), we obtain zeros because $C^i_{.jb} = 0$ and $L^i_{.jk}$ do not depend on y^k. So,

$$
R_{ja} = P^i{}_{jia} = 0.
$$

Taking

$$
S^a{}_{bcd} = \frac{\partial C^a_{.bc}}{\partial y^d} - \frac{\partial C^a_{.bd}}{\partial y^c} + C^e_{.bc}C^a_{.ed} - C^e_{.bd}C^a_{.ec}.
$$

from (30) and contracting the indices in order to obtain the Ricci coefficients,

$$
R_{bc} = \frac{\partial C^d_{.bc}}{\partial y^d} - \frac{\partial C^d_{.bd}}{\partial y^c} + C^e_{.bc}C^d_{.ed} - C^e_{.bd}C^d_{.ec}
$$

with $C^d_{.bd} = C_b$ already computed, see (122), we obtain

$$
R_{bc} = \left(C^4_{.bc}\right)^* - \partial_c C_b + C^4_{.bc}C_4 - C^4_{.b4}C^4_{.4c} - C^4_{.b5}C^5_{.4c} - C^5_{.b4}C^4_{.5c} - C^5_{.b5}C^5_{.5c}.
$$

There are nontrivial values,

$$
\begin{aligned}
R_{44} &= \left(C^4_{.44}\right)^* - C^*_4 + C^4_{44}(C_4 - C^4_{44}) - \left(C^5_{.45}\right)^2 \\
&= \beta^{**}_4 - (\beta^*_4 + \beta^*_5)^* + \beta^*_4\left(\beta^*_4 + \beta^*_5 - \beta^*_4\right) - (\beta^*_5)^* \\
R_{55} &= \left(C^4_{.55}\right)^* - C^4_{.55}\left(-C_4 + 2C^5_{.45}\right) \\
&= -\left(\frac{h^*_5}{2h_4}\right)^* + \frac{h^*_5}{2h_4}\left(2\beta^*_5 + \beta^*_4 - \beta^*_5\right)
\end{aligned}
$$

Introducing

$$
\beta^*_4 = \frac{h^*_4}{2h_4}, \beta^*_5 = \frac{h^*_5}{2h_5}
$$

we get

$$
R^4_4 = R^5_5 = \frac{1}{2h_4h_5}\left[-h^{**}_5 + \frac{\left(h^*_5\right)^2}{2h_5} + \frac{h^*_4h^*_5}{2h_4}\right]
$$

which is just (65).

Theorem 4.1 is proven.

B Reductions from 5D to 4D

To construct a $5D \to 4D$ reduction for the ansatz (63) and (4) is to eliminate from formulas the variable x^1 and to consider a 4D space (parametrized by local coordinates (x^2, x^3, v, y^5)) being trivially embedded into 5D space (parametrized by local coordinates (x^1, x^2, x^3, v, y^5) with $g_{11} = \pm 1, g_{1\widehat{\alpha}} = 0, \widehat{\alpha} = 2,3,4,5)$ with possible 4D conformal and anholonomic transforms depending only on variables (x^2, x^3, v). We suppose that the 4D metric $g_{\widehat{\alpha}\widehat{\beta}}$ could be of arbitrary signature. In order to emphasize that some coordinates are stated just for a such 4D space we put "hats" on the Greek indices, $\widehat{\alpha}, \widehat{\beta}, ...$ and on the Latin indices from the middle of alphabet, $\widehat{i}, \widehat{j}, ... = 2,3$, where $u^{\widehat{\alpha}} = \left(x^{\widehat{i}}, y^a\right) = (x^2, x^3, y^4, y^5)$.

In result, the Theorems 4.1 and 4.2, Corollaries 4.1 and 4.2 and Theorem 4.3 can be reformulated for 4D gravity with mixed holonomic–anholonomic variables. We outline here the most important properties of a such reduction.

- The metric (62) with ansatz (63) and metric (69) with (4) are respectively transformed on 4D spaces to the values:

  The first type 4D off–diagonal metric is taken

$$\mathbf{g} = \mathbf{g}_{\widehat{\alpha}\widehat{\beta}} \left(x^{\widehat{i}}, v\right) du^{\widehat{\alpha}} \otimes du^{\widehat{\beta}} \tag{123}$$

  with the metric coefficients $g_{\widehat{\alpha}\widehat{\beta}}$ parametrized

$$\begin{bmatrix} g_2 + {w_2}^2 h_4 + {n_2}^2 h_5 & w_2 w_3 h_4 + n_2 n_3 h_5 & w_2 h_4 & n_2 h_5 \\ w_2 w_3 h_4 + n_2 n_3 h_5 & g_3 + {w_3}^2 h_4 + {n_3}^2 h_5 & w_3 h_4 & n_3 h_5 \\ w_2 h_4 & w_3 h_4 & h_4 & 0 \\ n_2 h_5 & n_3 h_5 & 0 & h_5 \end{bmatrix}, \tag{124}$$

  where the coefficients are some necessary smoothly class functions of type:

$$\begin{aligned} g_{2,3} &= g_{2,3}(x^2, x^3), h_{4,5} = h_{4,5}(x^{\widehat{k}}, v), \\ w_{\widehat{i}} &= w_{\widehat{i}}(x^{\widehat{k}}, v), n_{\widehat{i}} = n_{\widehat{i}}(x^{\widehat{k}}, v);\ \widehat{i}, \widehat{k} = 2,3. \end{aligned}$$

  The anholonomically and conformally transformed 4D off–diagonal metric is

$$\mathbf{g} = \omega^2(x^{\widehat{i}}, v) \hat{\mathbf{g}}_{\widehat{\alpha}\widehat{\beta}} \left(x^{\widehat{i}}, v\right) du^{\widehat{\alpha}} \otimes du^{\widehat{\beta}}, \tag{125}$$

  were the coefficients $\hat{\mathbf{g}}_{\widehat{\alpha}\widehat{\beta}}$ are parametrized by the ansatz

$$\begin{bmatrix} g_2 + ({w_2}^2 + {\zeta_2}^2) h_4 + {n_2}^2 h_5 & (w_2 w_3 + +\zeta_2 \zeta_3) h_4 + n_2 n_3 h_5 & (w_2 + \zeta_2) h_4 & n_2 h_5 \\ (w_2 w_3 + +\zeta_2 \zeta_3) h_4 + n_2 n_3 h_5 & g_3 + ({w_3}^2 + {\zeta_3}^2) h_4 + {n_3}^2 h_5 & (w_3 + \zeta_3) h_4 & n_3 h_5 \\ (w_2 + \zeta_2) h_4 & (w_3 + \zeta_3) h_4 & h_4 & 0 \\ n_2 h_5 & n_3 h_5 & 0 & h_5 + \zeta_5 h_4 \end{bmatrix} \tag{126}$$

  where $\zeta_{\widehat{i}} = \zeta_{\widehat{i}} \left(x^{\widehat{k}}, v\right)$ and we shall restrict our considerations for $\zeta_5 = 0$.

- We obtain a quadratic line element

$$\delta s^2 = g_2(dx^2)^2 + g_3(dx^3)^2 + h_4(\delta v)^2 + h_5(\delta y^5)^2, \tag{127}$$

written with respect to the anholonomic co–frame $\left(dx^{\widehat{i}}, \delta v, \delta y^5\right)$, where

$$\delta v = dv + w_{\widehat{i}} dx^{\widehat{i}} \text{ and } \delta y^5 = dy^5 + n_{\widehat{i}} dx^{\widehat{i}} \tag{128}$$

is the dual of $(\delta_{\widehat{i}}, \partial_4, \partial_5)$, where

$$\delta_{\widehat{i}} = \partial_{\widehat{i}} + w_{\widehat{i}} \partial_4 + n_{\widehat{i}} \partial_5. \tag{129}$$

- If the conditions of the 4D variant of the Theorem 4.1 are satisfied, we have the same equations (76) –(79) were we substitute $h_4 = h_4\left(x^{\widehat{k}}, v\right)$ and $h_5 = h_5\left(x^{\widehat{k}}, v\right)$. As a consequence we have $\alpha_i\left(x^k, v\right) \to \alpha_{\widehat{i}}\left(x^{\widehat{k}}, v\right), \beta = \beta\left(x^{\widehat{k}}, v\right)$ and $\gamma = \gamma\left(x^{\widehat{k}}, v\right)$ resulting in $w_{\widehat{i}} = w_{\widehat{i}}\left(x^{\widehat{k}}, v\right)$ and $n_{\widehat{i}} = n_{\widehat{i}}\left(x^{\widehat{k}}, v\right)$.

- The 4D line element with conformal factor (127) subjected to an anhlonomic map with $\zeta_5 = 0$ transforms into

$$\delta s^2 = \omega^2(x^{\widehat{i}}, v)[g_2(dx^2)^2 + g_3(dx^3)^2 + h_4(\hat{\delta} v)^2 + h_5(\delta y^5)^2], \tag{130}$$

given with respect to the anholonomic co–frame $\left(dx^{\widehat{i}}, \hat{\delta} v, \delta y^5\right)$, where

$$\delta v = dv + (w_{\widehat{i}} + \zeta_{\widehat{i}}) dx^{\widehat{i}} \text{ and } \delta y^5 = dy^5 + n_{\widehat{i}} dx^{\widehat{i}} \tag{131}$$

is dual to the frame $\left(\hat{\delta}_{\widehat{i}}, \partial_4, \hat{\partial}_5\right)$ with

$$\hat{\delta}_{\widehat{i}} = \partial_{\widehat{i}} - (w_{\widehat{i}} + \zeta_{\widehat{i}})\partial_4 + n_{\widehat{i}} \partial_5, \hat{\partial}_5 = \partial_5. \tag{132}$$

- The formulas (71) and (73) from Theorem 4.2 must be modified into a 4D form

$$\hat{\delta}_{\widehat{i}} h_4 = 0 \text{ and } \hat{\delta}_{\widehat{i}} \omega = 0 \tag{133}$$

and the values $\zeta_{\widehat{i}} = (\widehat{\zeta_i}, \zeta_5 = 0)$ are found as to be a unique solution of (71); for instance, if

$$\omega^{q_1/q_2} = h_4 \ (q_1 \text{ and } q_2 \text{ are integers}),$$

$\zeta_{\widehat{i}}$ satisfy the equations

$$\partial_{\widehat{i}} \omega - (w_{\widehat{i}} + \zeta_{\widehat{i}})\omega^* = 0. \tag{134}$$

- One holds the same formulas (83)-(88) from the Theorem 4.3 on the general form of exact solutions with that difference that their 4D analogs are to be obtained by reductions of holonomic indices, $\widehat{i} \to i$, and holonomic coordinates, $x^i \to x^{\widehat{i}}$, i. e. in the 4D solutions there is not contained the variable x^1.

- The formulae (74) for the nontrivial coefficients of the Einstein tensor in 4D stated by the Corollary 4.1 are written

$$G_2^2 = G_3^3 = -S_4^4, G_4^4 = G_5^5 = -R_2^2. \tag{135}$$

- For symmetries of the Einstein tensor (135), we can introduce a matter field source with a diagonal energy momentum tensor, like it is stated in the Corollary 4.2 by the conditions (75), which in 4D are transformed into

$$\Upsilon_2^2 = \Upsilon_3^3 = \Upsilon_2(x^2, x^3, v), \ \Upsilon_4^4 = \Upsilon_5^5 = \Upsilon_4(x^2, x^3). \tag{136}$$

The 4D dimensional off–diagonal ansatz may model certain generalized Lagrange configurations and Lagrange–affine solutions. They can also include certain 3D Finsler or Lagrange metrics but with 2D frame transforms of the corresponding quadratic forms and N–connections.

C Generalized Lagrange–Affine Spaces

We outline and give a brief characterization of five classes of generalized Finsler–affine spaces (contained in the Table 1 from Ref. [6]; see also in that work the details on classification of such geometries). We note that the N–connection curvature is computed following the formula $\Omega_{ij}^a = \delta_{[i}N_{j]}^a$, see (5), for any N–connection N_i^a. A d–connection $\mathbf{D} = [\mathbf{\Gamma}_{\beta\gamma}^{\alpha}] = [L^i{}_{jk}, L^a{}_{bk}, C^i{}_{jc}, C^a{}_{bc}]$ defines nontrivial d–torsions $\mathbf{T}^{\alpha}{}_{\beta\gamma} = [L^i_{[\ jk]}, C^i{}_{ja}, \Omega_{ij}^a, T^a{}_{bj}, C^a{}_{[bc]}]$ and d–curvatures $\mathbf{R}^{\alpha}{}_{\beta\gamma\tau} = [R^i{}_{jkl}, R^a{}_{bkl}, P^i{}_{jka}, P^c{}_{bka}, S^i{}_{jbc}, S^a{}_{dbc}]$ adapted to the N–connection structure (see, respectively, the formulas (29) and (30)). Any generic off–diagonal metric $g_{\alpha\beta}$ is associated to a N–connection structure and reprezented as a d–metric $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ (see formula (11)). The components of a N–connection and a d–metric define the canonical d–connection $\mathbf{D} = [\widehat{\mathbf{\Gamma}}_{\beta\gamma}^{\alpha}] = [\widehat{L}^i{}_{jk}, \widehat{L}^a{}_{bk}, \widehat{C}^i{}_{jc}, \widehat{C}^a{}_{bc}]$ (see (27)) with the corresponding values of d–torsions $\widehat{\mathbf{T}}^{\alpha}{}_{\beta\gamma}$ and d–curvatures $\widehat{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau}$. The nonmetricity d–fields are computed by using formula $\mathbf{Q}_{\alpha\beta\gamma} = -\mathbf{D}_{\alpha}\mathbf{g}_{\beta\gamma} = [Q_{ijk}, Q_{iab}, Q_{ajk}, Q_{abc}]$, see (13).

The Table 1 outlines five classes of geometries modeled in the framework of metric–affine geometry as spaces with nontrivial N–connection structure (for simplicity, we omitted the Berwald configurations, see Ref. [6]).

1. Metric–affine spaces (in brief, MA) are those stated as certain manifolds V^{n+m} of necessary smoothly class provided with arbitrary metric, $g_{\alpha\beta}$, and linear connection, $\Gamma_{\beta\gamma}^{\alpha}$, structures. For generic off–diagonal metrics, a MA space always admits nontrivial N–connection structures. Nevertheless, in general, only the metric field $g_{\alpha\beta}$ can be transformed into a d–metric one $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$, but $\Gamma_{\beta\gamma}^{\alpha}$ can be not adapted to the N–connection structure. As a consequence, the general strength fields $\left(T_{\beta\gamma}^{\alpha}, R^{\alpha}{}_{\beta\gamma\tau}, Q_{\alpha\beta\gamma}\right)$ can be also not N–adapted.

2. Distinguished metric–affine spaces (DMA) are defined as manifolds $\mathbf{V}^{n+m}$ provided with N–connection structure N_i^a, d–metric field (11) and arbitrary d–connection $\mathbf{\Gamma}_{\beta\gamma}^{\alpha}$. In this case, all strenghs $\left(\mathbf{T}^{\alpha}{}_{\beta\gamma}, \mathbf{R}^{\alpha}{}_{\beta\gamma\tau}, \mathbf{Q}_{\alpha\beta\gamma}\right)$ are N–adapted.

3. Generalized Lagrange–affine spaces (GLA), $\mathbf{GLa}^n = (V^n, g_{ij}(x,y),\ {}^{[a]}\mathbf{\Gamma}^{\alpha}{}_{\beta})$, are modeled as distinguished metric–affine spaces of odd–dimension, $\mathbf{V}^{n+n}$, provided with generic off–diagonal metrics with associated N–connection inducing a tangent bundle structure. The d–metric $\mathbf{g}_{[a]}$ and the d–connection ${}^{[a]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \left({}^{[a]}L^{i}_{jk},\ {}^{[a]}C^{i}_{jc}\right)$ are similar to those for the usual Lagrange spaces but with distorsions ${}^{[a]}\ \mathbf{Z}^{\alpha}{}_{\beta}$ inducing general nontrivial nonmetricity d–fields ${}^{[a]}\mathbf{Q}_{\alpha\beta\gamma}$.

4. Lagrange–affine spaces (LA), $\mathbf{La}^n = (V^n, g^{[L]}_{ij}(x,y),\ {}^{[b]}\mathbf{\Gamma}^{\alpha}{}_{\beta})$, are provided with a Lagrange quadratic form $g^{[L]}_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$ inducing the canonical N–connection structure ${}^{[cL]}\mathbf{N} = \{\ {}^{[cL]}N^i_j\}$ for a Lagrange space $\mathbf{L}^n = (V^n, g_{ij}(x,y))$ but with a d–connection structure ${}^{[b]}\mathbf{\Gamma}^{\gamma}{}_{\alpha} = {}^{[b]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}$ distorted by arbitrary torsion, $\mathbf{T}_{\beta}$, and nonmetricity d–fields, $\mathbf{Q}_{\beta\gamma\alpha}$, when ${}^{[b]}\mathbf{\Gamma}^{\alpha}{}_{\beta} = {}^{[L]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta} + {}^{[b]}\ \mathbf{Z}^{\alpha}{}_{\beta}$. This is a particular case of GLA spaces with prescribed types of N–connection ${}^{[cL]}N^i_j$ and d–metric to be like in Lagrange geometry.

5. Finsler–affine spaces (FA), $\mathbf{Fa}^n = \left(V^n, F\left(x,y\right),\ {}^{[f]}\mathbf{\Gamma}^{\alpha}{}_{\beta}\right)$, in their turn are introduced by further restrictions of $\mathbf{La}^n$ to a quadratic form $g^{[F]}_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ constructed from a Finsler metric $F\left(x^i, y^j\right)$. It is induced the canonical N–connection structure ${}^{[F]}\mathbf{N} = \left\{\ {}^{[F]}N^i_j\right\}$ as in the Finsler space $\mathbf{F}^n = (V^n, F\left(x,y\right))$ but with a d–connection structure ${}^{[f]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}$ distorted by arbitrary torsion, $\mathbf{T}^{\alpha}_{\beta\gamma}$, and nonmetricity, $\mathbf{Q}_{\beta\gamma\tau}$, d–fields, ${}^{[f]}\mathbf{\Gamma}^{\alpha}{}_{\beta} = {}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta} + {}^{[f]}\ \mathbf{Z}^{\alpha}{}_{\beta}$,where ${}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta\gamma}$ is the canonical Finsler d–connection.

Space	N–connection/ N–curvature metric/ d–metric	(d–)connection/ (d-)torsion	(d-)curvature/ (d–)nonmetricity
1. MA	N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	$\Gamma_{\beta\gamma}^{\alpha}$ $T_{\ \beta\gamma}^{\alpha}$	$R_{\ \beta\gamma\tau}^{\alpha}$ $Q_{\alpha\beta\gamma}$
2. DMA	N_i^a, Ω_{ij}^a $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	$\mathbf{\Gamma}_{\beta\gamma}^{\alpha}$ $\mathbf{T}_{\ \beta\gamma}^{\alpha}$	$\mathbf{R}_{\ \beta\gamma\tau}^{\alpha}$ $\mathbf{Q}_{\alpha\beta\gamma}$
3. GLA	$\dim i = \dim a$ N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{[a]} = [g_{ij}, h_{kl}]$	${}^{[a]}\mathbf{\Gamma}_{\ \alpha\beta}^{\gamma}$ ${}^{[a]}\mathbf{T}_{\ \beta\gamma}^{\alpha}$	${}^{[a]}\mathbf{R}_{\ \beta\gamma\tau}^{\alpha}$ ${}^{[a]}\mathbf{Q}_{\alpha\beta\gamma}$
4. LA	$\dim i = \dim a$ ${}^{[cL]}N_j^i$, ${}^{[cL]}\Omega_{ij}^a$ d–metr.$\mathbf{g}_{\alpha\beta}^{[L]}$	${}^{[b]}\mathbf{\Gamma}_{\ \alpha\beta}^{\gamma}$ ${}^{[b]}\mathbf{T}_{\ \beta\gamma}^{\alpha}$	${}^{[b]}\mathbf{R}_{\ \beta\gamma\tau}^{\alpha}$ ${}^{[b]}\mathbf{Q}_{\alpha\beta\gamma} = -\ {}^{[b]}\mathbf{D}_{\alpha}\mathbf{g}_{\beta\gamma}^{[L]}$
5. FA	$\dim i = \dim a$ ${}^{[F]}N_j^i$; ${}^{[F]}\Omega_{ij}^k$ d–metr.$\mathbf{g}_{\alpha\beta}^{[F]}$	${}^{[f]}\mathbf{\Gamma}_{\ \alpha\beta}^{\gamma}$ ${}^{[f]}\mathbf{T}_{\ \beta\gamma}^{\alpha}$	${}^{[f]}\mathbf{R}_{\ \beta\gamma\tau}^{\alpha}$ ${}^{[f]}\mathbf{Q}_{\alpha\beta\gamma} = -\ {}^{[f]}\mathbf{D}_{\alpha}\mathbf{g}_{\beta\gamma}^{[F]}$

Table 1: Generalized Finsler/Lagrange–affine spaces

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